

# Nonstandard Analysis and Generalized Functions

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*Abstract:* This application of nonstandard analysis utilizes the notion of the highly-saturated enlargement. These nonstandard methods are applied to the theory of generalized functions (distributions) and demonstrates how such analysis clarifies many aspects of this theory.

## 1. Additional Modeling Concepts.

In what follows, the basic notation and definitions are as they appear in [3]. However, except as mentioned in the appendix of [3], a significant aspect of nonstandard analysis has not been developed fully. The superstructure constructed in [3] is a model for  $\Gamma$  where  $\Gamma$  is the set of all sentences that hold in the structure. Since it is constructed from  $\mathbb{R}$ , it is called a model (for real analysis), for apparently every true statement from analysis holds true in the superstructure. The elementary nonstandard structure  ${}^*\mathcal{M} = ({}^*\mathcal{H}, \in, =)$  is associated with the standard model  $\mathcal{M} = (\mathcal{H}, \in, =)$  in a slightly special sense. Due to the applications in [3], it was not necessary to discuss  ${}^*\mathcal{M}$  relative to its special properties. This is no longer the case.

The structure  ${}^*\mathcal{M}$  is constructed from a (bounded) ultrapower based upon the structure  $\mathcal{M}$ . [1, p. 15–19], [2], [5, p. 83–88] It is assumed that there are constants that denote every member of  $\mathcal{H}$  where we do not differentiate between a constant and the object it names. Suppose that  $J$  is the index set and that  $\mathcal{U}$  is an appropriate ultrafilter on  $J$ . Let infinite  $A \in \mathcal{H}$  and  $A$  contains no individuals in  $\mathbb{R}$ . Let  $f \in \mathcal{H}^J$  and  $f(j) = A$ , for each  $j \in J$ . This is the constant map, constant in two ways both as a mathematical entity and relative to the value being denoted by a constant in our language. Some authors define an injection  $k$ , which is denoted by  $e$  in [5] and  $i$  in [1], such that  $k(A) = [A]$ , where  $[A]$  is the  $\mathcal{U}$ -equivalence class in the ultrapower that contains the constant map  $f$ . To obtain an isomorphic copy of  $\mathcal{M}$ , we could follow the usual Mostowski collapsing process as described in [5, p. 84–85] that gives the  $*$  mapping and let this map be restricted to the domain of all the constant sequence  $\mathcal{U}$ -equivalence classes in the ultrapower. This leads to an isomorphic copy of  $\mathcal{M}$ . [5, p 85.] But, care must be taken relative to the interpretation of the  $*$  map. It must always be remembered the  ${}^*\mathcal{M}$  is a model of the bounded expressions that hold in  $\mathcal{M}$  although sometimes the bounding set is not expressly stated in an expression it must be understood that the quantifiers are

restricted to specific elements in some  $X_n$  for  $\mathcal{M}$  and for the corresponding  ${}^*X_n$  for  ${}^*\mathcal{H}$ . The members of  ${}^*X_n$  are “internal” entities.

Consider the first-order statement  $a \in b$  about objects in  $\mathcal{H}$ . Then the statement  ${}^*a \in {}^*b$  holds in  ${}^*\mathcal{M}$  and is relative to the embedding of  $\mathcal{M}$  into  ${}^*\mathcal{M}$ . Now consider a set  $A \in \mathcal{H}$  and the bounded statement  $S = \forall x((x \in A) \wedge (x \in b))$ , which is in the required form, and let  $S' = \{x \mid (x \in A) \wedge (x \in b)\}$ . Then  $S$  holds in  $\mathcal{M}$  if and only if  ${}^*S = \forall x((x \in {}^*A) \wedge (x \in {}^*b))$  holds in  ${}^*\mathcal{M}$ . This indicates that the quantifiers are restricted to members of  ${}^*A$ . Since  $A$  is a set,  $A \in X_n$  for some  $n \geq 1$ . Then from Proposition 1 (iv) [3], it follows that each specific “ $x$ ” that satisfies  ${}^*S'$  is a member of  ${}^*X_0$  or  ${}^*X_n$ , where  $n \geq 1$ . Thus each such “ $x$ ” is a member of  ${}^*\mathcal{H}$ .

On the other hand, if  $S$  holds in  $\mathcal{M}$ , then  $S$  also holds in the isomorphic copy of  $\mathcal{M}$ , where the quantifier is restricted to constant sequence  $\mathcal{U}$ -equivalence classes. Hence the only members in the isomorphic copy of the set  $S'$  are the  ${}^*a$  such that  $({}^*a \in {}^*A) \wedge ({}^*a \in {}^*b)$ . But is this isomorphic copy of  $S'$  a member of  ${}^*\mathcal{H}$ ? One of the ideas behind the concept of a nonstandard structure is that if  $S'$  is infinite, then its isomorphic copy is not a member of  ${}^*\mathcal{H}$  although it is a subset of  ${}^*A$  and  ${}^*A \in {}^*\mathcal{H}$ . In this case, the set  ${}^*S'$  contains entities that are not produced by the  ${}^*$  map. In order to discuss sets such as the isomorphic copy of  $S'$ , a second superstructure  $(\mathcal{Y}, \in, =)$  is generated with ground set  $Y_0 = {}^*X_0$ . It is within this superstructure that we can specifically construct the sets that restrict the quantifiers when bounded statements are interpreted for the isomorphic copy of  $\mathcal{M}$ .

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**Definition 1.1 (Sigma Operator, Standard Copy)** If a set  $A \in \mathcal{H}$ , then  $\sigma A = \{{}^*a \mid a \in A\}$ , where the  ${}^*a$  in this form denotes the constant sequence  $\mathcal{U}$ -equivalence class [8, p. 44-45].

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Let  $\sigma\mathcal{H} = \{\sigma A \mid A \in \mathcal{H}\}$ . There is a certain confusion of symbols that one tries to avoid. When the isomorphic embedding is being considered, it is understood that  ${}^*a$  means something different when viewed as a *set* of objects. The symbol  ${}^*a$  is used only as a name for the equivalence class in the ultrapower that contains the constant sequence for the isomorphic embedding. But, relative to the structure  ${}^*\mathcal{M}$ , and as a nonempty set,  ${}^*a$  contains the objects in  $\sigma a$ . Thus relative to the sets in  $(\mathcal{Y}, \in, =)$ , the isomorphic copy is determined by the  $\sigma$  and  ${}^*$  operators. It is this fact that has lead to a minimizing of the use of the  $\sigma$  notation. Thus it is customary to do our real analysis in  $\mathcal{M}$  rather than in  $\sigma\mathcal{M} = (\sigma\mathcal{H}, \in, =)$  knowing that, when *comparisons* are to be made, we can apply the isomorphism to obtain the actual objects in  $\sigma\mathcal{H}$  that are used for modeling purposes. I mention that significant fact is that if a set  $A \in \mathcal{H}$  is finite, then  ${}^*A = \sigma A$ . One might say that the map  $\sigma$  preserves the hierarchy of finite sets. (Some authors find no need to consider  $\sigma\mathcal{M}$  [5].)

From the viewpoint of abstract model theory, what are the real numbers? The real numbers is considered to be ANY structure isomorphic to the “standard” structure. The standard structure is considered to be a nonempty set  $\mathbb{R}$  with the appropriate operators and unique elements defined and such that the operators,

unique elements and subsets of  $\mathbb{R}$  satisfy a set of axioms. Under the above isomorphism  $*$ , Theorem 3.1.1 in [3] implies that the isomorphic copy of the real numbers can be considered as THE real numbers and real analysis takes place in  ${}^\sigma\mathcal{H}$ . Thus, as has become customary, we let  ${}^\sigma\mathbb{R} = \mathbb{R}$ . Moreover, to more fully express this identification of the real numbers, consider how this isomorphism deals with the operator  $\subset$ . From the definition of the operator  $\subset$ , it follows that if  $A \in \mathbb{R}$ , then  ${}^\sigma A = \emptyset$ . Further, given two  $A, B \in \mathcal{H}$ . Then  $A \subset B$  if and only if  ${}^\sigma A \subset {}^\sigma B$ . Hence, if  $A \subset \mathbb{R}$ , then  $A = {}^\sigma A \subset {}^*A$ . Basic operators such as  $+$ ,  $\cdot$ ,  $<$ , under the isomorphism, become the operators  ${}^\sigma+$ ,  ${}^\sigma\cdot$ ,  ${}^\sigma<$  which then become THE operators  $+$ ,  $\cdot$ ,  $<$  for the field  $\mathbb{R}$ . Although the  ${}^\sigma$  notation could continue to be used on sets at any point when one is in doubt, it has become customary to remove this notation in some cases like  ${}^\sigma\mathbb{R} = \mathbb{R}$ . Often relations such as  ${}^*+$ ,  ${}^*\cdot$ ,  ${}^*<$  for the ordered field  ${}^*\mathbb{R}$  are considered as the actual extensions of the relations  $+$ ,  $\cdot$ ,  $<$ .

Relative to notation, what this means is that many of the constants in  $C(\mathcal{H})$  that denote the members of  $\mathcal{H}$  will now be used to denote many members of  ${}^\sigma\mathcal{H}$ . New symbols  ${}^\sigma A$  are used to denote other members of  ${}^\sigma\mathcal{H}$ . Then we have objects in  ${}^*\mathcal{H}$  that have names in the symbol set  $C({}^*\mathcal{H})$  and represent the internal extended standard objects. These are denoted by use of the  $*$  notation. All other members of  ${}^*\mathcal{H}$ , other than the internal members, are denoted by distinctly different symbols in  $C({}^*\mathcal{H})$ . Since there are only so many symbols that can be used, we must state that this non-“started” symbol represents an “internal” object. Each object in  $(\mathcal{V}, \in, =)$  has a symbol name. All such objects that are not denoted by any previously defined notation must be explicitly defined as “external” objects. In this regard, if infinite  $A \in \mathcal{H}$ , then for the actual structure  ${}^*\mathcal{M}$  soon to be constructed, the object (denoted by)  ${}^\sigma A$  is external.

Relative to mathematical modeling, if we are modeling physical entities with names taken from a specific discipline dictionary, then it is immaterial which real analysis structure is used for the modeling correspondence. Thus, the isomorphic  ${}^\sigma\mathcal{M}$  is chosen as the appropriate structure. Hence, if the Euclidean n-space function  $\mathbf{v}(t)$  corresponds to a natural physical world (i.e. N-world) vector, then the function  ${}^*\mathbf{v}(t)$  corresponds to a nonstandard physical world (i.e. NSP-world) vector. Since  $\mathbf{v} \subset {}^*\mathbf{v}$ , and  $\mathbf{v}, {}^*\mathbf{v} \in (\mathcal{V}, \in, =)$ , the N-world physical vector can be assumed to be a restriction of the NSP-world vector to the N-world. Although more can be said about the effects of such a restriction relative to direct and indirect observation, it is not necessary, at this point, to delve more deeply into such concepts.

There has also arisen a certain terminology. Suppose that  $f$  is a continuous map from  $\mathbb{R}$  into  $\mathbb{R}$  (i.e.  $f: \mathbb{R} \rightarrow \mathbb{R}$ ). Then one can write that  ${}^*f$  is a  $*$ -continuous map from  ${}^*\mathbb{R}$  to  ${}^*\mathbb{R}$ . The term “ $*$ -continuous” is often replaced in scientific discourse by the term “hypercontinuous.” On the other hand, some authors leave the term in this form and when communicating orally say “hypercontinuous” or “star-continuous.”

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**Definition 1.2 (Concurrent Relation)** A (bounded) binary relation  $\Phi$  in  $\mathcal{H}$  and, hence, in  ${}^\sigma\mathcal{H}$ , is *concurrent* if the following holds. For each finite  $\neq \emptyset$  set  $A = \{(a_1, b_1), \dots, (a_n, b_n)\} \subset \Phi$  there exists in the range,  $R(\Phi)$ , of  $\Phi$  some  $b$  such that  $\{(a_1, b), \dots, (a_n, b)\} \subset \Phi$ .

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Now as to the actual construction of the nonstandard structure  ${}^*\mathcal{M}$ , a special ultrafilter is selected which has the following *enlarging* property.

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**Definition 1.3 (Enlargement)** The structure  ${}^*\mathcal{M}$  is an *enlargement* if for every concurrent relation  $\Phi$  with the domain  $D(\Phi)$  there exists an internal  $b \in {}^*(R(\Phi)) = R({}^*\Phi)$  such that for each  ${}^*a \in {}^\sigma D(\Phi)$ ,  $({}^*a, b) \in {}^*\Phi$ .

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In all that follows, it is assumed that  ${}^*\mathcal{M}$  is, at the least, an enlargement. Also note that all members of  ${}^*(R(\Phi))$  are internal.

**Theorem 1.1.** *Let infinite  $A \in \mathcal{H}$ . Then  ${}^\sigma A \subset {}^*A$  and  ${}^\sigma A \neq {}^*A$ .*

Proof. Suppose that infinite  $A \in \mathcal{H}$ . Consider the relation  $\Phi = \{(x, y) \mid (x \in A) \wedge (y \in B) \wedge (x \neq y)\}$ . Suppose that  $\{(a_1, b_1), \dots, (a_n, b_n)\} \subset \Phi$ . However, since  $A$  is infinite, there exists some  $b \in A$  such that  $b \neq a_i$ ,  $i = 1, \dots, n$ . Hence,  $\{(a_1, b), \dots, (a_n, b)\} \subset \Phi$ . Thus  $\Phi$  is a concurrent relation. Consequently, there is an internal  $b \in R({}^*\Phi)$  such that  $({}^*a, b) \in {}^*\Phi$  and  $b \neq {}^*a$ . This completes the proof.

Often just one identified concurrent relation can determine a major portion of an entire nonstandard theory. A few more examples indicate this fact. First, consider the extension of the absolute value function to  ${}^*\mathbb{R}$ . By definition, for any  $r \in \mathbb{R}$  if  $r \geq 0$ , then  $|r| = r$  and for any  $r \in \mathbb{R}$  if  $r < 0$ , then  $|r| = -r$ . Stated formally, we have that  $S = \forall x(((x \in \mathbb{R}) \wedge (r \geq 0)) \rightarrow (|r| = r)) \wedge ((x \in \mathbb{R}) \wedge (r < 0)) \rightarrow (|r| = -r)$ . This statement holds in  $\mathcal{M}$ . Thus its  ${}^*$ -transform holds in  ${}^*\mathcal{M}$ . The  ${}^*$ -transform is  ${}^*S = \forall x(((x \in {}^*\mathbb{R}) \wedge (r \geq 0)) \rightarrow ({}^*|r| = r)) \wedge ((x \in {}^*\mathbb{R}) \wedge (r < 0)) \rightarrow ({}^*|r| = -r)$ , where  $|\cdot|$  is viewed as a unary operator. Thus the operator  ${}^*|\cdot|$  is but the absolute value operator as it is defined for the totally ordered field  ${}^*\mathbb{R}$ . Hence we can drop the  ${}^*$  notation from  ${}^*|\cdot|$ .

**Theorem 1.2.** *There exists in  ${}^*\mathbb{R}$ , a nonzero infinitesimal.*

Proof. Consider the relation  $\Phi = \{(n, m) \mid (0 < (1/m) < (1/n)) \wedge (n \in \mathbb{N}) \wedge (m \in \mathbb{N})\}$ . Suppose that  $\{(n_1, m_1), \dots, (n_j, m_j)\} \subset \Phi$ . Let  $M = \max\{m_1, \dots, m_j\}$ . Then  $\{(n_1, M+1), \dots, (n_j, M+1)\} \subset \Phi$  implies that  $\Phi$  is concurrent. Hence, there exists some  $\Lambda \in {}^*\mathbb{N}$  such that  $0 < 1/\Lambda < 1/n$  for all  $n \in {}^\sigma\mathbb{N} = \mathbb{N}$ . Now consider any positive  $r \in \mathbb{R}$ . Then there exists some  $n \in \mathbb{N}$  such that  $0 < 1/n < r$ . Hence,  $|1/\Lambda| < r$ . But since  $r$  is an arbitrary positive real number, this last results hold for all  $r \in \mathbb{R}^+$ . This completes the proof.

An examination of chapter 2 in [3] shows that the properties of the set of all infinitesimals  $\mu(0)$  are determined from Theorem 1.2. One of the most significant portions of the nonstandard theory of analysis is relative to the set of all “hyperfinite” sets. These are all of the internal sets  $A \in {}^*(F(\mathcal{H}) = \cup\{{}^*(F(X_n)) \mid X_n \in \mathcal{H}\})$ . Also note that if  $A \in X_n$ , ( $n > 0$ ), then  $F(A) \in X_{n+1}$ . Hence,  ${}^*(F(A)) \in {}^*X_{n+1}$ . Viewed as a mapping, we have that  ${}^*F$  is defined on all internal sets in  ${}^*\mathcal{H}$  and if  $B \in \mathcal{H}$ , then  ${}^*(F(B)) = {}^*F({}^*B)$ . As shown in [3] Theorem 4.3.2, the above definition for the hyperfinite sets is equivalent to the internal bijection definition. The hyperfinite sets satisfy in  ${}^*\mathcal{M}$  all of the first-order properties associated with finite sets. But from the exterior viewpoint of nonstandard analysis, such sets are far from being finite.

To get the full strength of nonstandard analysis as it relates to generalized functions, we need to concept of the  $\kappa$ -saturated enlargement, where  $\kappa$  is a infinite cardinal number.

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**Definition 1.4 (  $\kappa$ -Saturated )** The structure  ${}^*\mathcal{M}$  is a  $\kappa$ -saturated if given any internal (bounded) binary relation  $\Phi$ , with the internal domain  $D(\Phi)$ , that is concurrent on  $A \subset D(\Phi)$ , where cardinality of  $A < \kappa$ , then there exists an internal  $b \in R(\Phi)$ , the internal range of  $\Phi$ , such that for each  $a \in A$ ,  $(a, b) \in \Phi$ .

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Throughout the remaining portions of this paper, we assume that  ${}^*\mathcal{M}$  is a  $\kappa$ -saturated, where  $\kappa$  is any (regular) cardinal number greater than the cardinality of  $\mathcal{H}$ . This would also imply that  ${}^*\mathcal{M}$  is an enlargement. It can be shown by means of the ultralimit process, such (bounded)  $\kappa$ -saturated enlargements exist.

**Theorem 1.3.** Consider internal  $B$  and any  $A \subset B$  such that cardinality of  $A < \kappa$ . Then there exists a hyperfinite set  $\Omega$  such that  $A \subset \Omega \subset B$ .

Proof. From the construction of  $\mathcal{H}$  we know that there is some  $X_n$ ,  $n \geq 1$  and  $B \in {}^*X_n$ ,  ${}^*F(B) \in {}^*X_n$ , if  $q \in B$ ,  $\{q\} \in X_n$  and  $q \in {}^*X_0 \cup {}^*X_{n-1}$ . Consider the internal binary relation  $\Phi = \{(x, y) \mid (x \in y \in {}^*F(B)) \wedge (x \in {}^*X_0 \cup {}^*X_{n-1}) \wedge (y \in X_n)\}$ . Suppose that  $\{(a_1, b_1), \dots, (a_j, b_j)\} \subset \Phi$ . By  ${}^*$ -transfer of the standard theorem, it follows that  $b = b_1 \cup \dots \cup b_j \in {}^*F(B)$  and  $a_i \in b$ ,  $i = 1, \dots, j$ . Hence,  $\Phi$  is a concurrent on its domain. But  $A \subset D(\Phi)$  and has the appropriate cardinality. Hence, there exists some  $\Omega \in {}^*(F(B))$  such that  $A \subset \Omega \subset B$ . This completes the proof.

**Corollary 1.3.1.** Consider standard  $A$ . Then there exists a hyperfinite set  $\Omega$  such that  ${}^\sigma A \subset \Omega \subset {}^*A$ .

Proof. Simply note that the cardinality of  $A$  is less than  $\kappa$ .

The hyperfinite sets are the basic building blocks of the nonstandard theory of probability spaces.

## 2. Generalized Functions.

The functions considered are real valued functions. It is not difficult to extend all of the results in this section to complex valued functions. Further all standard functions map  $\mathbb{R}$  into  $\mathbb{R}$ . Let  $C^\infty$  be the set of all real valued functions defined on  $\mathbb{R}$  which have derivatives of all orders at each  $x \in \mathbb{R}$ . The set  ${}^*C^\infty$  contains some very interesting  ${}^*$ -continuous and  ${}^*$ -differentiable functions. Throughout this paper, nonempty  $\mathcal{D} \subset C^\infty$  is always the notation for what is called the *test space*. Each member of  $\mathcal{D}$  must be a function with bounded support. This implies that if  $g \in \mathcal{D}$ , then there is some  $c \in \mathbb{R}$  such that  $g(x) = 0$  for all  $|x| \geq c$ .

Usually one is interested in the generation of linear functionals. The customary generating functions are maps from  $\mathbb{R}$  into  $\mathbb{R}$ . The basic method of generation is by integration. Usually, the customary integration is Lebesgue integration although it appears the generalized Riemann integral can also be used. The reason that Lebesgue is useful is that this integral applies to many highly discontinuous standard functions, has useful convergence properties and, operationally, is sufficient. For our purposes, the Lebesgue integral is considered as an operator in the sense that it is 3-tuple with the first coordinate a function, the second an interval (or for other applications a measurable subset of  $\mathbb{R}$ ), and the third coordinate the value when it exists.

Our customary standard generating functions,  $CS$ , have the property that they are Lebesgue measurable on  $[a, b]$ , for  $a \leq b$ ,  $a, b \in \mathbb{R}$  and the integral  $\int_a^b (f(x))^2 dx \in \mathbb{R}$  (i.e.  $f \in \mathcal{L}^2([c, d])$  a classical Banach Space). If  $f$  is measurable and bounded on  $[a, b]$  and  $f \in \mathcal{L}([a, b])$  then  $f \in \mathcal{L}^2([a, b])$ . [7, p. 219] It is known that if  $f \in \mathcal{L}^2(E)$ , and for the Lebesgue measure,  $m$ ,  $m(E) \in \mathbb{R}$ , then  $f \in \mathcal{L}(E)$ . [7, p. 220] From this, it follows that  $\int_{-\infty}^{\infty} f(x)g(x) dx \in \mathbb{R}$  for each  $g \in \mathcal{D}$ . Our functions are restricted to members of internal set  $\cap \{ {}^*\mathcal{L}^2([c, d]) \mid (c \leq d) \wedge (c, d \in {}^*\mathbb{R}) \}$  so that the  ${}^*$ -transform of the classical Schwarz inequality applies. For the purposes of this paper, the set of internal generating functions is has a slightly different formation and can only be assumed to be an external subset of  $\cap \{ {}^*\mathcal{L}^2([c, d]) \mid (c \leq d) \wedge (c, d \in {}^*\mathbb{R}) \}$ . Recall that  $\mathcal{O}$  is the set of all limited numbers in  ${}^*\mathbb{R}$ . Note that this set is also called the set of finite numbers. It is clear that if  $f$  is a customary standard function, then for each  $c \leq d$ ,  $c, d \in {}^*\mathbb{R}$ ,  $\int_c^d ({}^*f(x))^2 dx \in {}^*\mathbb{R}$ . Thus  $\int_c^d ({}^*f(x))^2 dx \in {}^*\mathbb{R}$  when  $c, d \in \mathcal{O}$ . Further,  $\int_{-\infty}^{\infty} f(x)g(x) dx = r$  implies that  $\int_{-\infty}^{\infty} {}^*f(x) {}^*g(x) dx = {}^*r = r$ . First, are there members of  $CS$  such that  $\int_a^b ({}^*f(x))^2 dx \in \mathcal{O}$  when  $c, d \in \mathcal{O}$ ?

**Theorem 2.1.** *Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ . Suppose that  $f$  is measurable (in the Lebesgue sense) and bounded on all intervals  $[a, b]$ ,  $a \leq b$ ,  $a, b \in \mathbb{R}$ . Then for each  $c \leq d$ ,  $(c, d \in {}^*\mathbb{R})$ ,  $\int_c^d {}^*f(x) dx \in {}^*\mathbb{R}$  and  $\int_c^d ({}^*f(x))^2 dx \in {}^*\mathbb{R}$  and if  $c, d \in \mathcal{O}$ , then  $\int_c^d {}^*f(x) dx \in \mathcal{O}$  and  $\int_c^d ({}^*f(x))^2 dx \in \mathcal{O}$ .*

*Proof.* It follows from the hypotheses, that  $f \in \mathcal{L}([a, b])$  and for each interval  $[a, b]$  there exists some  $M \in \mathbb{R}$  such that  $|f(x)| < M$ , for each  $x \in [a, b]$ . By  ${}^*$ -transfer,  $\int_c^d {}^*f(x) dx \in {}^*\mathbb{R}$  for  $c \leq d$ ,  $(c, d \in {}^*\mathbb{R})$ . Let  $c \leq d$  and  $c, d \in \mathcal{O}$ . Then

there exist  $a, b \in \mathbb{R}$  such that  $c \approx a$ ,  $d \approx b$ . There are four cases to consider but one will suffice as a prototype. Suppose that  $c \leq a$ ,  $d \leq b$ . Let  $m$  denote the Lebesgue measure on the measurable subsets of  $\mathbb{R}$ . For each  $x, y \in {}^*\mathbb{R}$  such that  $x \leq y$ ,  ${}^*m([x, y]) = y - x$  by  ${}^*$ -transfer. Hence,  ${}^*m([c, a]) = a - c \approx 0$ ,  ${}^*m([d, b]) = b - d \approx 0$ . Now  $\int_c^b {}^*f dx = \int_c^a {}^*f dx + \int_a^d {}^*f dx + \int_d^b {}^*f dx$  by  ${}^*$ -transfer. Consider  $\int_c^a {}^*f dx$ . There exists some  $g \in \mathbb{R}$  such that  $g \leq c$ . Hence  $[c, b] \subset [g, b]$ . There exists some  $M \in \mathbb{R}$  such that  $|f(x)| \leq M$  for each  $x \in [c, b]$ . Again by  ${}^*$ -transfer  $|{}^*f(x)| \leq M$  for each  $x \in {}^*[c, b]$ . By  ${}^*$ -transfer,

$$-M({}^*m([c, a])) = -M(a - c) \leq \int_c^a {}^*f dx \leq M({}^*m([c, a])) = M(a - c).$$

Consequently,  $\int_c^a {}^*f dx \approx 0$ . In like manner,  $\int_d^b {}^*f dx \approx 0$ . Therefore,  $\int_c^d {}^*f dx \approx {}^*\int_a^b {}^*f dx = r \in \mathbb{R}$ . Hence,  $\int_c^d {}^*f(x) dx \in \mathcal{O}$ .

For the second part, simply consider the known standard result that if  $f \in \mathcal{L}([a, b])$  and  $g$  is bounded and measurable on  $[a, b]$ , then  $fg \in \mathcal{L}([a, b])$  and apply the first part. This completes the proof.

**Corollary 2.1.1.** *Let continuous  $f: \mathbb{R} \rightarrow \mathbb{R}$ . Then for each  $c \leq d$ , ( $c, d \in {}^*\mathbb{R}$ ),  $\int_c^d {}^*f(x) dx \in {}^*\mathbb{R}$  and  $\int_c^d ({}^*f(x))^2 dx \in {}^*\mathbb{R}$  and if  $c, d \in \mathcal{O}$ , then  $\int_c^d {}^*f(x) dx \in \mathcal{O}$  and  $\int_c^d ({}^*f(x))^2 dx \in \mathcal{O}$ .*

Proof. Clearly,  $f \in \mathcal{L}([a, b])$  and  $f \in \mathcal{L}^2([a, b])$ . The same proof as theorem 2.1 yields the conclusions.

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**Definition 2.1 (Generalized Functions)** Let  $T$  be the set of internal functions such that for each  $f \in T$ , (i)  $f: {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$  and (ii)  $\int_c^d (f(x))^2 dx \in {}^*\mathbb{R}$  for each pair  $c, d \in \mathcal{O}$ ,  $c \leq d$ , and (iii)  $\int_{-\infty}^{\infty} f(x) {}^*g(x) dx \in \mathcal{O}$  for each  $g \in \mathcal{D}$ .

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A function  $f$  such that  $f^2$  is a limited integral over limited intervals, will be said to have the *limited* (ii) property.

**Theorem 2.2.** *Suppose that internal  $f: {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$ , has the limited (ii) property, then  $f \in T$ .*

Proof. From the hypotheses,  $f$  satisfies (i) and (ii) of Definition 2.1. We need only show that  $f$  satisfies (iii). We know that for  $g \in \mathcal{D}$  there is some  $c \in \mathbb{R}^+$  such that  $g(x) = 0$  for all  $x \in \mathbb{R}$  such that  $|x| \geq c$ . Then by Schwarz's inequality (in concise notation)

$$\begin{aligned} \left( \int_{-\infty}^{\infty} f {}^*g \right)^2 &= \left( \int_{-c}^c f {}^*g \right)^2 \leq \\ &\left( \int_{-c}^c f^2 \right) \left( \int_{-c}^c {}^*g^2 \right). \end{aligned}$$

But since  $g$  is continuous on  $[-c, c]$ ,  $\left(\int_{-c}^c g^2\right) = r \in \mathbb{R}$  implies that  $\left(*\int_{-c}^c *g^2\right) = r \in \mathcal{O}$ . Since  $f$  has the limited (ii) property, then

$$\left(*\int_{-c}^c f^2\right) \left(*\int_{-c}^c g^2\right) = h \in \mathcal{O}.$$

Consequently,  $*\int_{-c}^c f *g \in \mathcal{O}$  and the proof is complete.

**Example 2.1.** Does  $T$  contain nonextended standard functions? Let  $0 \neq \epsilon \approx 0$  (i.e.  $\epsilon \in \mu(0)$ , where  $\mu(0)$  is the set of infinitesimals.) Define the function  $f = \{(x, y) \mid (x \in {}^*\mathbb{R}) \wedge (y \in {}^*\mathbb{R}) \wedge (y = \epsilon)\}$ . From the internal definition principle [3],  $f$  is an internal  $*$ -constant function. It is not the extension of a standard function since  $\epsilon \notin \mathbb{R}$ . Moreover,  $*\int_c^d f^2 = \epsilon^2(d - c) \in \mu(0) \subset \mathcal{O}$  for all  $c, d$ , ( $c \leq d$ ) such that  $c, d \in \mathcal{O}$ . By Theorem 2.2,  $f \in T$ . Also consider the internal function  $f_1 = \{(x, y) \mid (x \in {}^*\mathbb{R}) \wedge (y \in {}^*\mathbb{R}) \wedge (y = *\sin(x + \epsilon))\}$ . Now  $f_1^2$  is  $*$ -continuous on  ${}^*\mathbb{R}$  and, hence,  $*$ -integrable on any  $[c, d]$ , ( $c \leq d$ ),  $c, d \in \mathcal{O}$ . Further,  $-1 \leq *\sin^2(x + \epsilon) \leq 1$  for all  $x \in {}^*\mathbb{R}$ . Hence,  $-1(d - c) \leq *\int_c^d *\sin^2 \leq 1(d - c)$  for all limited  $c \leq d$  implies that  $*\int_c^d *\sin^2 \in \mathcal{O}$  for all limited  $c \leq d$ . Again by Theorem 2.2,  $f_1 \in T$ .

**Theorem 2.3** *The set  $\sigma(CS) \subset T$ .*

Proof. From the discussion prior to Theorem 2.1.

From Theorem 2.1, Corollary 2.1.1, and by Theorem 2.2,  $T$  contains many significant extended standard functions, among others, that will be useful later in this investigation. The next result also indicates that one of the conclusions is sufficient for an internal function to be a member of  $T$ .

The next definition indicates why (iii) in Definition 2.1 is significant.

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**Definition 2.2 (The Quasi-Inner Product)** Let  $f \in T$  and  $g \in \mathcal{D}$ . Define  $\langle f, *g \rangle = *\int_{-\infty}^{\infty} f(x) *g(x) dx$ . Note that  $\langle \cdot, \cdot \rangle$  is an ordered pair notation since the elements come from possibly different sets.

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### 3. Some Abstract Algebra.

For most applications of the theory of generalized functions, it may be assumed or a function can be appropriately redefined so that the standard function being considered is at the least bounded on the closed intervals. Call  $[c, d]$ , where  $c \leq d$  and  $c, d \in {}^*\mathbb{R}$  a *limited  $*$ -closed interval*. Recall that if  $f \in \mathcal{L}^2(E)$  for measurable  $E$  such that  $m(E) \in \mathbb{R}$ , then  $f \in \mathcal{L}(E)$ . [7, p. 220] For the internal functions that concern us,  $*$ -transfer says that if  $f \in {}^*\mathcal{L}^2([c, d])$ , ( $c \leq d$ ,  $c, d \in {}^*\mathbb{R}$ ), then  $f \in {}^*\mathcal{L}[c, d]$ . This last statement certainly holds for the limited  $*$ -closed intervals. Recall the standard result that if  $f \in \mathcal{L}([a, b])$ , and  $g$  is measurable and bounded



on  $[a, b]$ , then  $fg \in \mathcal{L}([a, b])$  [7, p. 219]. Thus by  $*$ -transfer for internal function  $f$  that is  $*$ -bounded on a limited  $*$ -closed interval  $[c, d]$ , the function  $f \in {}^*\mathcal{L}^2([c, d])$  if and only if  $f \in {}^*\mathcal{L}([c, d])$ . As shown in the next theorem, in some special cases, the set  $T$  is closed under multiplication of functions.

**Theorem 3.1** *The following algebraic properties hold.*

(a) *The set  $T$  is an unitary  $\mathcal{O}$ - module (left and right) over the set of limited numbers  $\mathcal{O}$  and  $T$  is a linear space over  $\mathbb{R}$ .*

(b) *If  $f, h \in T$  and  $f, h$  are  $*$ -bounded in limited  $*$ -closed intervals and the product  $fh$  has the limited (ii) property, then  $fh \in T$ .*

(c) *For the set of all continuous functions defined on  $\mathbb{R}$ ,  $C(\mathbb{R})$ ,  ${}^\sigma C(\mathbb{R}) \subset T$*

(d) *The set  ${}^\sigma \mathcal{D}$  is an ideal in ring with unity  ${}^\sigma C^\infty$ .*

(e) *If  $f \in T$  and  ${}^*g \in {}^\sigma C^\infty$ , then  $f{}^*g \in T$ .*

(f) *The the real valued operator  $\langle \cdot, \cdot \rangle$  is linear with respect to the field  $\mathbb{R}$  in the first and second coordinates. The standard part of  $\langle \cdot, \cdot \rangle$  is an inner product on  ${}^\sigma \mathcal{D}$ . Proof.*

(a) From the  $*$ -transfer of the known properties of  $\mathcal{L}^2([c, d])$ ,  $T$  is closed under function addition. Since  $\mathcal{O}$  is a ring, if  $\lambda \in \mathcal{O}$ , then  $\lambda f \in T$ . The functions  ${}^*\mathbf{1} \equiv 1$ ,  ${}^*\mathbf{0} \equiv 0 \in T$ . Hence,  $T$  is a unitary  $\mathcal{O}$ -module over the ring  $\mathcal{O}$ .

(b) A standard result says that if  $f \in \mathcal{L}^2([a, b])$  and  $g \in \mathcal{L}^2([a, b])$ , then  $fg \in \mathcal{L}([a, b])$ . By  $*$ -transfer, this statement holds for the limited  $*$ -closed intervals. If internal  $f$  and internal  $g$  are  $*$ -bounded on a limited  $*$ -closed interval, then internal  $fg$  is  $*$ -bounded on a limited  $*$ -closed interval. From our discussion prior to the statement of Theorem 3.1, in this case,  $fg \in {}^*\mathcal{L}([c, d])$ . But the  $*$ -measurable internal function  $fg$  is  $*$ -bounded on limited  $*$ -closed intervals. Thus  $fg \in {}^*\mathcal{L}^2([c, d])$  for the limited  $*$ -closed intervals. Further,  $(fg)^2$  is  $*$ -bounded on  $[c, d]$ . Obviously,  $fh$  satisfies (i) and, from the hypothesis, (ii) of Definition 2.1. Now by Theorem 2.2, it follows that  $fh \in T$ .

(c)  ${}^\sigma C(\mathbb{R}) \subset {}^\sigma(CS) \subset T$ .

(d) The sum and product of members of  $C^\infty$  with bounded support have bounded support and  $\mathbf{0} \in \mathcal{D}$ . Hence  ${}^\sigma \mathcal{D}$  is a subring of  ${}^\sigma C^\infty$ . From the bounded support property, if  $h \in C^\infty$  and  $g \in \mathcal{D}$ , then  $gh$  has bounded support. Thus  ${}^\sigma \mathcal{D}$  is an ideal in  ${}^\sigma C^\infty$ .

(e) Suppose you have a standard function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f \in \mathcal{L}^2([a, b])$  on standard interval  $[a, b]$ . Since  $h^2 \in C^\infty$  is bounded and measurable on  $[a, b]$ ,  $(fh)^2 \in \mathcal{L}([a, b])$  by Theorem 22.4s in [6, p. 127.]. Thus, by  $*$ -transfer, for  $f \in T$  and  ${}^*h \in {}^\sigma C^\infty$ ,  $f{}^*h$  satisfies (i) and (ii) of Definition 2.1. From (d), if  $g \in \mathcal{D}$ , then  $hg \in \mathcal{D}$ . Consequently,  ${}^*\int_{-\infty}^{\infty} f{}^*h{}^*g \in \mathcal{O}$  for each  $g \in \mathcal{D}$ .

(f) Only the basic algebra for members of  $T$ , where it is defined, needs to be verified. Let  $f \in T$ ,  $\lambda \in \mathbb{R}$ . We know that if  $f, h \in T$ ,  $f + h \in T$  and  $\lambda f, \lambda h \in T$ . If  ${}^*g_1, {}^*g_2 \in {}^\sigma \mathcal{D}$ ,  ${}^*g_1 + {}^*g_2 \in {}^\sigma \mathcal{D}$  and  $\lambda{}^*g_1, \lambda{}^*g_2 \in {}^\sigma \mathcal{D}$ . For the

second coordinate,  $\lambda\langle f, {}^*g_1 + {}^*g_2 \rangle = \lambda \int_{-\infty}^{\infty} f({}^*g_1 + {}^*g_2) = \int_{-\infty}^{\infty} \lambda f {}^*g_1 + \lambda f {}^*g_2 = \int_{-\infty}^{\infty} \lambda f {}^*g_1 + \int_{-\infty}^{\infty} \lambda f {}^*g_2 = \lambda \langle f, {}^*g_1 \rangle + \lambda \langle f, {}^*g_2 \rangle$ . In like manner for the first coordinate and, from the above,  $\lambda\langle f, g \rangle = \langle \lambda f, g \rangle = \langle f, \lambda g \rangle$ . The standard part operator is linear over  $\mathbb{R}$ . Hence, the composition of  $\langle \cdot, \cdot \rangle$  and  $\text{st}(\cdot)$  is linear over  $\mathbb{R}$ . Moreover, this composition yields a member of  $\mathbb{R}$ . Now  $\langle \cdot, \cdot \rangle$  is defined on all members of  ${}^\sigma\mathcal{D}$  independent of order. Further, if  $g, h \in {}^\sigma\mathcal{D}$ , then  $\langle g, h \rangle = \langle h, g \rangle \in \mathbb{R}$  implies that  $\text{st}(\langle g, h \rangle) = \text{st}(\langle h, g \rangle)$ , and  $\langle g, g \rangle \geq 0$  implies that  $\text{st}(\langle g, g \rangle) \geq 0$ . Since  $\mathcal{D}$  contains only continuous functions (with bounded support),  $\langle g, g \rangle = 0$  if and only if  $g = \mathbf{0}$ . Thus on  ${}^\sigma\mathcal{D}$  the operator  $\text{st}(\langle \cdot, \cdot \rangle)$  is an inner product.

#### 4. Functionals on $T$ .

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**Definition 4.1 (The Functional)** Let fixed  $f \in T$ . Then, for each  ${}^*g \in {}^\sigma\mathcal{D}$ , define  $f[g] = \text{st}(\langle f, {}^*g \rangle)$ . Let  $T_0 = \{f \mid (f \in T) \wedge \forall x((x = {}^*g) \wedge ({}^*g \in {}^\sigma\mathcal{D}) \rightarrow (\text{st}(\langle f, x \rangle) = 0))\}$ .

---

From the fact that the standard part operator is linear over  $\mathbb{R}$ , it follows from Theorem 3.1 (f), that  $f[\cdot]$  is a linear functional on  ${}^\sigma\mathcal{D}$ . Let  $\mathcal{F} = \{f[\cdot] \mid f \in T\}$ . Obviously there is a surjection  $\Phi: T \rightarrow \mathcal{F}$ . Since  $T$  is a linear space over  $\mathbb{R}$  and  $\Phi(\mathbf{0}) = \mathbf{0}[g] = 0$  for each  $g \in {}^\sigma\mathcal{D}$  and preserves scalar products and “sums,”  $\Phi$  is a vector space homomorphism.

**Theorem 4.1.** *The function  $f \in T_0$  if and only if  $\langle f, {}^*g \rangle \approx 0$  for all  $g \in \mathcal{D}$ .*

Proof. This comes from the fact that  $\text{st}(a) = 0$  if and only if  $a \approx 0$ .

**Theorem 4.2** *If  $f \in T$ , and  $\int_c^d (f(x))^2 dx \approx 0$  for all limited\*-closed intervals, then  $f \in T_0$ .*

Proof. This is the same as the proof of Theorem 2.2.

Obviously, more than one  $f \in T$  can yield the zero functional and  $T_0 \neq \emptyset$  since  $\mathbf{0} \in T_0$ . Now  $T_0$  is the kernel for this homomorphism, and, as is-known, the quotient linear space  $T/T_0$  is isomorphic to  $\mathcal{F}$ . Each element in  $T/T_0$  is an equivalence class of members of  $T$ . Then for  $h, g \in T$ , it follows that  $f, h \in \alpha \in T/T_0$  if and only if  $h - f \in T_0$  if and only if  $h[g] = f[g] = 0$  for all  ${}^*g \in {}^\sigma\mathcal{D}$ . It is this isomorphism that allows us to correspond a subset of  $T/T_0$  to all of the Schwarz generalized functions.

**Example 4.1.** Does  $T$  contain functions that yield the Dirac property? Let

$$b(t) = \begin{cases} \exp(-1/(1-t^2)), & |t| < 1 \\ 0, & \text{elsewhere} \end{cases}$$

This is a version of Cauchy’s flat function and it is known that  $b \in \mathcal{D}$ . We now compress this function. Let  $0 < \epsilon \in \mu(0)$  (a positive infinitesimal). Let  $c(t) = b(t/\epsilon)$ .

By  $^*$ -transfer,  $c \in {}^*C^\infty$  with support  $[-\epsilon, \epsilon]$ . We can normalize  $c$  by letting  $k = \int_{-\infty}^{\infty} c(t) dt \neq 0$ , and writing  $d(t) = (1/k)c(t)$ . Obviously,  $d \in {}^*C^\infty$ , is nonnegative, and  $\int_{-\infty}^{\infty} d(t) dt = 1$ . First, we show that  $d \in T$ . By  $^*$ -transfer,  $d$  is  $^*$ -bounded  $^*$ -measurable on each  $[c, d]$ ,  $c, d \in {}^*\mathbb{R}$ . Hence  $d \in {}^*\mathcal{L}^2([c, d])$ ,  $c, d \in \mathcal{O}$  (i.e.  $d \in {}^*(CS)$ ) and satisfies (i) and (ii) of Definition 2.1. We need only show that for all  $^*g \in {}^\sigma\mathcal{D}$ ,  $\langle d, ^*g \rangle \approx ^*g(0) = g(0)$ . Recall that the operators  $^*\max$  and the  $\max$  on  ${}^*\mathbb{R}$  are the same operator. By  $^*$ -transfer of the standard theorem,

$$|\int_{-\infty}^{\infty} d {}^*g| \leq \sup\{|^*g(t)| \mid t \in [-\epsilon, \epsilon]\} \cdot \int_{-\infty}^{\infty} d.$$

Note that  $\int_{-\epsilon}^{\epsilon} d = 1$ . The function  $|g|$  is continuous at 0. Hence, for each  $t \in \mu(0)$ ,  $|^*g(t)| \approx g(0)$ . From the  $^*$ -transfer of the extreme value theorem, there exists some  $t_1 \in [-\epsilon, \epsilon] \subset \mu(0)$  such that  $\sup\{|^*g(t)| \mid t \in [-\epsilon, \epsilon]\} = |^*g(t_1)|$  (i.e.  $\sup = \max$ ). Hence,  $\sup\{|^*g(t)| \mid t \in [-\epsilon, \epsilon]\} \in \mu(g(0)) \subset \mathcal{O}$ . Thus  $d \in T \cap {}^*(CS)$ . In a similar manner, noting that  $d$  is nonnegative, we have we have that  $g(0) \approx \min\{|^*g(t)| \mid t \in [-\epsilon, \epsilon]\} \leq \langle d, ^*g \rangle \leq \max\{|^*g(t)| \mid t \in [-\epsilon, \epsilon]\} \approx g(0)$ . Consequently, the functional  $d[g] = g(0)$  for all  $^*g \in {}^\sigma\mathcal{D}$ . But this last statement is the “shifting” property of Dirac when viewed as a  $^*$ -Lebesgue integration over  ${}^*\mathbb{R}$ . The same method shows that for any positive  $n \in \mathbb{N}$ ,  $d^n[g] = g(0)$ .

---

There are infinitely many internal functions in  ${}^*C^\infty$  that are in  $T$  and that determine the Dirac functional. In the standard theory, no such standard function exists and “something” is only symbolically introduced relative to the required shifting property. This yields what are called “singular” generalized functions. From the nonstandard viewpoint, at least for the  $d[\cdot]$ , such a concept of “singular” is no longer meaningful.

---

Since  $T/T_0$  is isomorphic to  $\mathcal{F}$ , then each  $f \in T$  such that  $f[g] = g(0)$  for all  $g \in \mathcal{D}$  are in the same member of  $T/T_0$ . We call this the *Dirac delta* equivalence class and denote it by  $\delta$ . Note that  $d^n \in \delta$ .

**Example 4.2.** The set  ${}^*\mathcal{D} \not\subset T$ . Consider the function  $b$  of example 4.1. Then  $0 < \inf\{d(t) \mid -1/2 \leq t \leq 1/2\} = d(1/2) \in \mathbb{R}$ . Further,  $0 < \int_{-\infty}^{\infty} d = r \in \mathbb{R}$ . As pointed out,  $b \in \mathcal{D}$ . Let  $\Lambda \in \mathbb{N}_\infty$  (the infinite natural numbers). Then by  $^*$ -transfer,  $\Lambda b \in {}^*\mathcal{D}$ . Now  $\inf\{\lambda {}^*d(t) \mid -1/2 \leq t \leq 1/2\} = \Lambda d(1/2) \in {}^*\mathbb{R} - \mathbb{R}$  and

$$\Lambda d(1/2) \left( \int_{-\infty}^{\infty} {}^*d \right) \leq {}^*\langle \lambda {}^*d, {}^*d \rangle.$$

Thus  ${}^*\langle \lambda {}^*d, {}^*d \rangle \notin \mathcal{O}$ .

---

**Definition 4.2 (Pre-generalized Functions)** Each member  $\alpha$  in the quotient linear space  $T/T_0$  is called a pre-generalized function and each member of  $T$  is a generalized function. From this point on, lower case Greek letters will always denote pre-generalized functions.

---

One of the reasons, the set  $T/T_0$  is called the set of pre-generalized functions is that a member of  $T/T_0$  need not correspond to a Schwarz generalized function. But before corresponding pre-generalized functions to Schwarz generalized functions, we have the following remarkable result first proved by Robinson. The functionals in  $\mathcal{F}$  are specifically generated by the  ${}^*\int_{-\infty}^{\infty}$ . Does this exhaust the entire collection of all linear functionals defined on  ${}^\sigma\mathcal{D}$ ? The following result shows the power of the enlargement concept.

**Theorem 4.3.** *Let  $\Delta$  represent any linear functional defined on  ${}^\sigma\mathcal{D}$ . Then there exists a  ${}^*$ -polynomial  $p_\Delta \in {}^*C^\infty \cap T$  such that  $\Delta = p_\Delta[\cdot] \in \mathcal{F}$ .*

Proof. First, let  $\Pi$  be the set of all polynomials defined on  $\mathbb{R}$  and  $\Delta: \mathcal{D} \rightarrow \mathbb{R}$ . Note that  $\Pi \subset C^\infty$ . Consider the binary relation  $R = \{((g, \Delta(g)), p) \mid (g \in \mathcal{D}) \wedge (\Delta(g) \in \mathbb{R}) \wedge (p \in \Pi) \wedge (\int_{-\infty}^{\infty} pg = \Delta(g))\}$ . What is needed is to show that  $R$  is concurrent on the domain  $\mathbb{R} \times \mathcal{D}$ .

Consider nonempty  $\mathcal{D}$  and a nonempty finite linear independent  $L = \{g_j \mid (j = 1, \dots, m) \wedge (1 \leq m)\} \subset \mathcal{D}$ . Then there exists some  $c > 0$  such that  $g_j(x) = 0$ ,  $i \leq j \leq m$ , and  $c$  can be selected so that each  $g_j$  is zero in a neighborhood of  $-c$  and  $c$ . From this we also have that for any  $f \in T$ ,

$$f[g_j] = \text{st}({}^*\int_{-c}^c f {}^*g_j) = a_j, \quad 1 \leq j \leq m. \quad (4.2.1)$$

Since each  $g_j \in \mathcal{D} \subset C^\infty$ , let each  $g_j$  be represented in terms of a series expansion of Legendre polynomials  $P_i$  where, using a simple transformation of the independent variable, the  $P_i$  have been extended to converge on  $[-c, c]$  rather than  $[-1, 1]$ . Hence,  $g_j(x) = \sum_{n=0}^{\infty} a_n^j P_n(x)$ , for all  $x \in [-c, c]$  and the convergence being uniform on any  $[-d, d]$ ,  $0 < d < c$ . We now use the method of the infinite matrix, a method used by Robinson and Bernstein to solve a specific case of the invariant subspace problem. From the linear independent assumption, the matrix

$$B = \begin{pmatrix} a_0^1 & a_1^1 & a_2^1 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ a_0^m & a_1^m & a_2^m & \cdots \end{pmatrix}$$

must be of rank  $m$ . Thus there is a finite collect of members  $B$  and, hence, of

subscripts  $0 \leq j_1 \leq \dots \leq j_m$  such that

$$A = \begin{pmatrix} a_{j_1}^1 & a_{j_2}^1 & a_{j_3}^1 & a_{j_m}^1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{j_1}^m & a_{j_2}^m & a_{j_3}^m & a_{j_m}^m \end{pmatrix}$$

and  $A$  is nonsingular. Now write

$$A^{-1} \begin{pmatrix} g_1(x) \\ \cdot \\ \cdot \\ \cdot \\ g_m(x) \end{pmatrix} = \begin{pmatrix} h_1(x) \\ \cdot \\ \cdot \\ \cdot \\ h_m(x) \end{pmatrix}.$$

Thus, we can write

$$h_l(x) = P_{j_l}(x) + k_l(x), \quad 1 \leq l \leq m,$$

where the Legendre polynomials in each  $k_l(x)$  do not contain any of the  $P_{j_l}(x)$ ,  $1 \leq l \leq m$ .

Now let

$$A^{-1} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}.$$

Consider the polynomial  $p(x) = c_1 P_{j_1} + \dots + c_m P_{j_m}(x)$ . We want to obtain the proper  $c_1, \dots, c_m$  such that  $p[g_j] = a_j$ ,  $1 \leq j \leq m$ . First, note that from the orthogonality of the Legendre polynomials  $\int_{-c}^c p k_l = 0$ ,  $1 \leq l \leq m$  and we know that  $\int_{-c}^c P_n^2 = r_n \neq 0$ . Select  $c_l = (1/r_l) b_l$ ,  $1 \leq l \leq m$ . Substituting this  $p$  with these coefficients into (4.2.1) yields

$$p[g_j] = \mathbf{st} \left( \int_{-c}^c {}^*p {}^*g_j \right) = \int_{-c}^c p g_j = a_j, \quad 1 \leq j \leq m. \quad (4.2.2)$$

Since every member of  $\mathcal{D} - L$  (if any) is a linear combination of the members of  $L$ , then it follows from the linearity of  $\Delta$  that for each  $g \in \mathcal{D}$ ,  $p[g] = \Delta(g)$ . Hence, the relation  $R$  is concurrent on  $\mathbb{R} \times \mathcal{D}$  and, hence on  ${}^\sigma \mathbb{R} \times {}^\sigma \mathcal{D}$ . Thus, there exists some  $p_\Delta \in {}^*\Pi \subset {}^*C^\infty$  such that for all  $(g, \Delta(g)) \in \mathbb{R} \times {}^\sigma \mathcal{D}$ ,  $((g, \Delta(g)), p_\Delta) \in {}^*R$ . Since  $\mathbb{R} \subset \mathcal{O}$ ,  $p_\Delta \in T$  and  $\int_{-\infty}^\infty p_\Delta {}^*g = \Delta(g)$ . Consequently,  $p_\Delta[g] = \Delta(g)$  and the proof is complete.

Theorem 4.3 is remarkable, since if  $\alpha \in T/T_0$ , there is  $p_\Delta \in {}^*\Pi$  such that  $p_\Delta \in \alpha$ . Furthermore,  $p_\Delta$  is a \*-finite sum of \*-Legendre polynomials. This means,

as difficult as it might be to imagine, there is a  $p_\Delta \in \delta$  such that  $p_\Delta[g] = g(0)$  for each  ${}^*g \in {}^\sigma\mathcal{D}$ . The internal function  $d$  used in Example 4.1 is NOT a member of  ${}^*\Pi$  by  ${}^*$ -transfer of the standard properties of the standard function  $c(t/a)$ ,  $0 < a$ . The function  $p_\Delta$  that generates every linear functional is also a member of  $T_R$ . Of course, Theorem 4.3 holds for other functions as well that are either simple modifications of the  $P_i$  or are such things as finite sums of trigonometric functions. **Note that every linear functional on  ${}^\sigma\mathcal{D}$  (i.e.  $\mathcal{D}$ ) can be generated by the Definition 4.1 process by means of  ${}^*$ -Riemann integration.** Although it is possible to remove many functions from each pre-generalized function  $\alpha$  by considering the quotient group formed from the sets  $P_0 = T \cap {}^*C^\infty$ , (or  $T \cap {}^*\Pi$ ) and  $p_1 = T_0 \cap {}^*C^\infty$ , (or  $T_0 \cap {}^*\Pi$ ), it is more significant to have such functions as  $d \in \delta$ . Later for a necessary simplification process, we will call a generalized function the function in  ${}^*C^\infty$  that exists by Theorem 4.3 and not consider the equivalence class at all.

## 5. Schwarz Generalized Functions.

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**Definition 5.1 (Schwarz Generalized Functions)** A  $f[\cdot] \in \mathcal{F}$  is a *Schwarz generalized function* if given any sequence  $\{g_n\} \subset \mathcal{D}$  such that

- (i) there exists some  $a \in \mathbb{R}$  such that for each  $n \in \mathbb{N}$ ,  $g_n(x) = 0$  for all  $|x| > a$ ,
- (ii) for each natural number  $k \geq 0$ ,  $g_n^{(k)}(x) \rightarrow 0$  uniformly on  $[-a, a]$ ,
- (iii) then  $f[g_n] \rightarrow 0$ .

Let  $\mathcal{D}'$  denote the set of Schwarz generalized functions.

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For every  $f[\cdot] \in \mathcal{D}'$ , there exists unique  $\alpha \in T/T_0$  under our isomorphism. Such an  $\alpha$  is called  $\mathcal{D}'$ -pre-generalized function and an  $f \in \alpha$  an  $\mathcal{D}'$ -generalized function. [The term *Schwarz generalized function* means the linear functions.] As previously pointed out,  ${}^\sigma(CS) \subset T$ . Using only  $k = 0$ , properties of Lebesgue integration, the Schwarz inequality, if  $f \in CS$ , then it follows that  ${}^*f[\cdot] \in \mathcal{D}'$ . Using the  ${}^*$ -transform of these previous properties, it follows, using the internal function  $d$  defined in Example 4.1, that  $d[\cdot] \in \mathcal{D}'$  although  $d$  is not an extended standard function. This shows the advantage of selecting specific members of a pre-generalized function.

**Theorem 5.1.** *Let  $f, h \in \alpha$ . Suppose that using the  ${}^*$ -transform of the definition of the derivative that  $f$  and  $h$  possess a  ${}^*$ -derivative  $f'$  and  $h'$  for each  $x \in {}^*\mathbb{R}$  and that  $f', h'$  are  ${}^*$ -continuous for each  $x \in {}^*\mathbb{R}$ . Then  $f', h' \in T$  and there exists a  $\beta$  such that  $f', h' \in \beta$ .*

Proof. By  ${}^*$ -transfer of the continuous case, it follows that  $f'$  satisfies (i), (ii)

of Definition 4.1. By  $*$ -integration by parts,  $\text{st}(\langle f', *g \rangle) = -\text{st}(\langle f, *g' \rangle) \in \mathbb{R}$  for each  $g \in \mathcal{D}$ . Since  $g' \in \mathcal{D}$ , then  $f' \in T$ . In like manner,  $h' \in T$ . Consider  $k = f - h$ . We know that  $\langle k, *g \rangle \approx 0$  for each  $g \in \mathcal{D}$ . Since  $k$  satisfies all the  $*$ -transformed derivative rules,  $\text{st}(\langle k', *g \rangle) = -\text{st}(\langle k, *g' \rangle) = 0$  for each  $g \in \mathcal{D}$ . Thus  $k' \in T_0$  by our remark after Theorem 4.2. Hence, there exists a unique  $\beta \in T/T_0$  such that  $f', h' \in \beta$ . This completes the proof.

**Corollary 5.1.1** *For every  $f[\cdot] \in \mathcal{F}$ , there exists a  $f'[\cdot] \in \mathcal{F}$  such that  $f'[g] = -f[g']$ .*

Since every pre-generalized function  $\alpha$  contains an internal  $f$  such that there is an internal function  $f'$  which is the  $*$ -continuous  $*$ -derivative of  $f$  on  $*\mathbb{R}$ , then to every pre-generalized function there corresponds a unique  $\beta$  such that  $f' \in \beta$ . We denote such a pre-generalized function by the notation  $\alpha'$ .

**Theorem 5.2.** *For every  $k \in \mathbb{N}$ , and every  $\alpha$ , there exists a unique  $\alpha^{(k)} \in T/T_0$ .*

Proof. Theorem 4.3 shows that for every  $\alpha \in T/T_0$  there exists an  $p \in {}^*C^\infty$  such that  $p \in \alpha$ . The result follows by induction and Theorem 5.1.

---

**Corollary 5.2.1** *For every  $0 \leq k \in \mathbb{N}$ , and for every  $f[\cdot] \in \mathcal{F}$ , there exists a  $f^{(k)}[\cdot] \in \mathcal{F}$  such that  $f^{(k)}[g] = (-1)^k f[g^{(k)}]$ .*

---

One can now see why included in the definition of the Schwarz generalized function is the additional part of (ii) for each  $k > 0$ . For, from above, we have the following important result.

**Theorem 5.3.** *If  $\alpha$  is a  $\mathcal{D}'$ -pre-generalized function, then  $\alpha^{(k)}$  is a  $\mathcal{D}'$ -pre-generalized function for each  $k \in \mathbb{N}$ .*

**Corollary 5.3.1.** *If  $f[\cdot] \in \mathcal{D}'$ , then, for each  $k \in \mathbb{N}$ , there exists a unique  $f^{(k)}[\cdot] \in \mathcal{D}'$  such that  $f^{(k)}[g] = (-1)^k f[g^{(k)}]$ .*

Although Corollary 5.3.1 is an important Schwarz generalized function result, the nonstandard theory is more general in that Corollary 5.2.1 holds.

## 6. “Continuity.”

Obviously, the definition of a Schwarz generalized function is designed to give the linear functional a type of continuity. In nonstandard analysis, there are various

types of continuity.

For two topological spaces  $(X, \tau)$  and  $(Y, \mathcal{T})$ , you always have the concept of a function  $f: X \rightarrow Y$  as being continuous at  $p \in X$  if for each  $G_1 \in \tau_1$ , such that  $f(p) \in G_1$ , there exists a  $G \in \tau$  such that  $p \in G$  and  $f(G) \subset G_1$ . In general, for standard  ${}^*p \in {}^\sigma X$ , the *topological monad* of  ${}^*p$  for a given topology  $\mathcal{T}$  is  $\mu_{\mathcal{T}}({}^*p) = \cap \{ {}^*G \mid p \in G \in \mathcal{T} \}$ . Then it can be shown that  $f$  is continuous at  $p \in X$  if and only if  ${}^*f(\mu_{\tau}({}^*p)) \subset \mu_{\mathcal{T}}({}^*f({}^*p))$ . Note that the reason we need to use the standard elements in the form  ${}^*p$  is that it is not assumed that  $X \cup Y$  are atoms within our set-theory. Let  ${}^*\mathbb{R}^+$  denote the set all positive hyperreal numbers.

---

**Definition 6.1 (Pseudo-metric Generated Space)** Given an internal set  $X$  and  $PM_X$  the internal set of all pseudo-metrics defined on  $X$ . If internal map  $\lambda \in PM_X$ , then for each  $x, y, z \in X$ , (i)  $\lambda(x, y) \in {}^*\mathbb{R}^+$ , (ii)  $\lambda(x, y) = \lambda(y, x)$ , (iii) if  $x = y$ , then  $\lambda(x, y) = 0$ , and (iv)  $\lambda(x, z) \leq \lambda(x, y) + \lambda(y, z)$ . Let nonempty  $\Lambda \subset PM_X$ . The entity  $(X, \Lambda)$  is an *pseudo-metric generated space*.

---

Each space  $(X, \Lambda)$  satisfies the  $*$ -transform of any general property for a pseudo-metric. To see this, note that there is some standard  $X_n$  such that  $X \in {}^*X_n$  and some standard set  $X_p$  such that internal  $PM_X \in {}^*X_p$ . Now, in general, for each  $X \in X_n$ , there exists a standard set  $PM_X$ . Thus there exists a standard set  $\mathcal{P} = \{PM_X \mid (PM_X \in X_p) \wedge (X \in X_n)\}$ . The internal sets of definition 6.1 are members of  ${}^*X_n$  and the internal  $PM_X \in {}^*\mathcal{P}$ . The defining property for members of internal  $PM_X$  is but the  $*$ -transform of the standard definition. Hence, using these sets, any general bounded first-order property about standard pseudo-metrics holds, by  $*$ -transfer, for members of an internal  $PM_X$ . For example, suppose that internal  $\lambda \in \Lambda$  is determined by an internal semi-norm  $\|\cdot\|_\lambda$  defined on an internal space  $X$  linear over  ${}^*\mathbb{R}$ . Then, for  $x, y \in X$ , we have that  $|\|x\|_\lambda - \|y\|_\lambda| \leq \|x - y\|_\lambda \leq \|x\|_\lambda + \|y\|_\lambda$ . Of course, our basic examples are the standard pseudo-metrics on a standard  $X$ .

**Example 6.1** Let  $SM$  be the set of all internal semi-norms defined on an internal linear space  $X$ . Thus  $\|\cdot\| \in SM$  if and only if for each  $x, q \in X$ , (i)  $\|q\| \in {}^*\mathbb{R}^+$ , (ii) for each  $\lambda \in {}^*\mathbb{R}$ ,  $\|\lambda q\| = |\lambda| \|q\|$ , (iii)  $\|x + q\| \leq \|x\| + \|q\|$ . Then defining internal  $\lambda: X \times X \rightarrow {}^*\mathbb{R}$  by  $\lambda(x, q) = \|x - q\|$  gives  $\lambda \in PM_X$ .



---

**Definition 6.2. (Monads about  $q \in X$ )** For  $(X, \Lambda)$  and  $q \in X$  the *monad about  $q$*  is  $\mu_\Lambda(q) = \{x \mid (x \in X) \wedge \forall \lambda((\lambda \in \Lambda) \rightarrow (\lambda(x, q) \in \mu(0)))\}$ , where  $\mu(0)$  is the set of infinitesimals in  ${}^*\mathbb{R}$ .

Let internal  $\lambda \in \Lambda$ . Then the  $\lambda$ -monad about  $q$  is  $\mu_\lambda(q) = \{x \mid (x \in X) \wedge (\lambda(x, q) \in \mu(0))\}$ . It is an important fact that  $\mu_\Lambda(q) = \cap\{\mu_\lambda(q) \mid \lambda \in \Lambda\}$ .

---

**Example 6.2** Let  $\mathcal{S}$  be a standard collection of pseudo-metrics defined on standard  $X$ . Consider the usual topology  $\mathcal{T}$ , generated by the subbase  $\mathcal{B}$  of all open balls determined by all the members of  $\mathcal{S}$  (i.e.  $B(p, \lambda, \epsilon)$ ,  $p \in X$ ,  $\lambda \in \mathcal{S}$ ,  $\epsilon \in \mathbb{R}^+$ ). For a topological space, a topological monad,  $\mu_{\mathcal{T}}({}^*p)$ , about standard  $p \in X$  is the set  $\cap\{{}^*G \mid p \in G \in \mathcal{T}\} = \cap\{{}^*G \mid {}^*p \in {}^*G \in {}^\sigma\mathcal{T}\}$  and, in general, is equal to  $\cap\{{}^*G \mid p \in G \in \mathcal{B}\}$  for any subbase  $\mathcal{B}$  for the topology. Thus under Definition 6.2, for  $p \in X$ , where  $\Lambda = {}^\sigma\mathcal{S}$ , the monad about  $p$ ,  $\mu_\Lambda({}^*p)$  is topological.

---

**Definition 6.3. ( $\approx$  and Monads)** When an infinitesimal relation  $\approx$  is defined on an internal  $X$ , this relation is, usually, an equivalence relation and is used to define a monad about each  $q \in X$ . The monad about  $q \in X$  is the equivalence class  $m_\approx(q) = \{x \mid x \approx q\}$ .

---

In general, the monad defined by 6.3 need not be the same as a monad as defined by a topology. But for Definition 6.2, an obvious equivalence relation does exist that correspond these monad concepts.

---

**Definition 6.4** For the space  $(X, \Lambda)$  and any  $x, y \in X$ , let  $x \approx y$  if and only if  $\forall \lambda((\lambda \in \Lambda) \rightarrow (\lambda(x, y) \in \mu(0)))$ . Also, for each  $\lambda \in \Lambda$ ,  $x \stackrel{\lambda}{\approx} y$  if and only if  $\lambda(x, y) \in \mu(0)$ .

---

It is immediate that the  $\approx$  [resp.  $\stackrel{\lambda}{\approx}$ ] of Definition 6.4 is an equivalence relation on  $X \times X$ , and that  $x \approx y$  [resp.  $x \stackrel{\lambda}{\approx} y$ ] if and only if  $x \in \mu_\Lambda(y)$  [resp.  $x \in \mu_\lambda(y)$ ] if and only if  $x \in m_\approx(y)$  [resp.  $x \in m_\lambda(y)$ .]

---

**Definition 6.5. (S-continuous)** For spaces  $(X, \Lambda)$  and  $(Y, \Pi)$  an internal  $f: X \rightarrow Y$  is *S-continuous* at  $q \in X$  if  $f(\mu_\Lambda(q)) \subset \mu_\Pi(f(q))$ . For pseudo-metric space  $(X, \lambda)$  and  $(Y, \pi)$ , S-continuity is defined for the spaces  $(X, \{\lambda\})$  and  $(X, \{\pi\})$ .

---

For an pseudo-metric,  $\lambda$  defined on internal  $X$ , you define, for each  $\epsilon \in {}^*\mathbb{R}^+$ , and for each  $q \in X$ , (in the usual way) the ball about  $q$  as  $B(q, \lambda, \epsilon) = \{x \mid (\lambda(x, q) <$

$\epsilon\}$ . Another type of continuity, that is usually restricted to standard spaces, is  $*$ -continuity. For a standard gauge space (i.e. the topological space generated by the set of pseudo-metrics  $\Lambda$ ), then the topology is generated by taking as a bases the finite intersection of standard balls. Since any finite set of real numbers has a minimum, using the neighborhood bases about a standard point, we have that for standard  $(X, \Lambda)$ ,  $(Y, \Pi)$  a standard functions  $f: X \rightarrow Y$  is continuous at  $p \in X$  if and only if for each  $\epsilon \in \mathbb{R}^+$  and each  $\pi \in \Pi$  there exists a  $\delta \in \mathbb{R}^+$  and a finite set of pseudo-metrics  $\lambda_i \in \Lambda$ ,  $1 \leq i \leq n$  such that whenever  $x \in X$  and  $\lambda_i(x, p) < \delta$  for each  $i$ ,  $1 \leq i \leq n$  then  $\pi(f(x), f(p)) < \epsilon$ . The  $*$ -transfer of this statement is used to define another type of continuity.

---

**Definition 6.6 ( $*$ -continuity)** Consider pseudo-metric generated spaces  $(X, \Lambda)$ ,  $(Y, \Pi)$ . An internal map  $f: (X, \Lambda) \rightarrow (Y, \Pi)$  is  $*$ -continuous at  $q \in X$  if for each  $\epsilon \in {}^*\mathbb{R}^+$  and each  $\pi \in \Pi$  there exists a  $\delta \in {}^*\mathbb{R}^+$  and a  $*$ -finite set  $\{\lambda_i \mid 1 \leq i \leq \omega\} \subset \Lambda$  such that whenever  $x \in X$  and  $\lambda_i(x, q) < \delta$  for each  $i$  such that  $1 \leq i \leq \omega$  then  $\pi(f(x), f(q)) < \epsilon$ . For internal metric spaces,  $(X, \lambda)$  and  $(Y, \pi)$ ,  $*$ -continuity is defined for  $(X, \{\lambda\})$  and  $(Y, \{\pi\})$ .

---

Notice that  $*$ -continuity is not defined solely in terms of the standard points in  $X$ . Also the specific  $\epsilon$  and  $\delta$  required for  $*$ -continuity are members of  ${}^*\mathbb{R}^+$ , not just the standard positive reals. For standard  $(X, \Lambda)$ ,  $(Y, \Pi)$ ,  $f: (X, \Lambda) \rightarrow (Y, \Pi)$  is continuous at  $p \in X$  if and only if  $*f: ({}^*X, {}^*\Lambda) \rightarrow ({}^*Y, {}^*\Pi)$  is  $*$ -continuous at  $*p$ .

**Example 6.4.** By  $*$ -transfer, for any infinite  $\Lambda \in \mathbb{N}_\infty$ , the function  $f(x) = {}^*\sin(\Lambda x)$  is  $*$ -continuous on  ${}^*\mathbb{R}$  with respect to the standard norm  $|\cdot|$ .

The  $*$ -continuous functions have an important place in theoretical physics [4] [See example 6.6 below]. But since they have all of the properties of the continuous functions, they are not considered “interesting” to some members of the mathematics community.

**Example 6.5.** Although the  $*$ -continuous function of example 6.4 is  $*$ -continuous at  $x = 0$ , it is S-discontinuous at  $x = 0$ . Let infinitesimal  $\epsilon = (1/\Lambda)(\pi/2)$ . Then for S-continuity we must have that  ${}^*\sin(\Lambda \cdot 0) = {}^*\sin(0) = 0 \approx {}^*\sin(\Lambda((1/\Lambda)(\pi/2))) = 1$ ; a contradiction.

**Example 6.6.** The function  $d$  of example 4.1 that generates the Dirac functional is  $*$ -continuous at  $x = 0$  but is S-discontinuous. Since the defining statement for  $d$  is the  $*$ -transform of the collection of all functions  $c$  formed by letting  $t \rightarrow t/a$ ,

where  $a$  is any nonzero real number, it follows that  $d$  is  $^*$ -continuous at all  $x \in ^*\mathbb{R}$  (i.e.  $d$  is a member of a set of  $^*$ -continuous functions with this property.) However,  $[-\epsilon, \epsilon] \subset d(\mu(0))$  and  $d[-\epsilon, \epsilon] = ^*[0, e^{-1}]$  imply that  $d$  is not S-continuous at  $x = 0$ .

The “S” in S-continuous means “standardly” in the sense that the approximating numbers  $\epsilon, \delta$  are standard numbers. One example shows what S-continuity is trying to accomplish.

**Example 6.7.** Let  $0 < \epsilon \in \mu(0)$  and  $a \in \mathbb{R}$ . Define on  $^*\mathbb{R}$

$$f(x) = \begin{cases} \epsilon + a, & x < 0 \\ a, & x \geq b \end{cases}$$

This is, by  $^*$ -transfer, an internal function that is  $^*$ -continuous for all nonzero  $x \in ^*\mathbb{R}$  and is  $^*$ -discontinuous at  $x = b$ . But,  $f$  is S-continuous at  $x = b$ . For take any positive  $c \in \mathbb{R}$ , then no matter what  $x \in ^*\mathbb{R}$  you select,  $|f(x) - a| < c$ . **That is the  $^*$ -discontinuity is so “small” that it is not “visible” in the standard world.**

For the topological spaces used in the theory of generalized functions, does the concept of S-continuous correspond to the concept described in Example 6.7? In order to examine the relation between S-continuity and  $^*$ -continuity for generalized functions, a slight diversion is necessary

For a given standard  $X$ , suppose  $\mathcal{V}$  is a collection of subsets of  $X$  such that  $\emptyset \notin \mathcal{V}$  and  $\mathcal{V}$  has the finite intersection property (i.e. the intersection of finitely many members is not the empty set). Then the collection  $\mathcal{V}$  is a *filter subbase* on  $X$ . Further, if there exists some  $q \in ^*X$  such that  $q \in ^*G$  for each  $G \in \mathcal{V}$ , then  $\mathcal{V}$  is, obviously, a filter subbase which is termed the *local filter subbase* at  $q$  and denoted by  $\mathcal{V}_q$ .

---

**Definition 6.7 (Monads of a Filter Subbase)** For standard  $X$ , let  $\mathcal{V}$  be a filter subbase (either local or otherwise) on  $X$ . Then let  $\mu_{\mathcal{V}} = \cap \{ ^*V \mid V \in \mathcal{V} \}$ . If  $\mathcal{V}$  is local at  $q \in ^*X$ , then since  $q \in \mu_{\mathcal{V}}$  the monad is written as  $\mu_{\mathcal{V}}(q)$ .

---

**Example 6.8** For any standard pseudo-metric space  $(X, \Lambda)$ , and  $p \in X$ , consider the set,  $\mathcal{B} = \{B(p, \lambda, \epsilon) \mid (\lambda \in \Lambda) \wedge (\epsilon \in \mathbb{R}^+)\}$ , of all balls about  $p$ . Then  $\mathcal{B}$  is a local filter subbase at  $p$ . For any filter subbase  $\mathcal{B}$ , let  $\langle \mathcal{B} \rangle$  be the set obtained by taking finite intersections of members of  $\mathcal{B}$ . Obviously,  $\mathcal{B} \subset \langle \mathcal{B} \rangle$  and  $\mu_{\mathcal{B}} = \mu_{\langle \mathcal{B} \rangle}$ .

**Theorem 6.2.** *Let  $X$  be standard set and  $\mathcal{V}$  any standard filter subbase on  $X$ . Then  $\mu_{\mathcal{V}} \neq \emptyset$ .*

Proof. We know that there is some  $X_n$ ,  $n \leq 1$ , such that  $\mathcal{V} \in X_n$ , and if  $x \in \mathcal{V}$ , then  $x \in X_n$ . Further, if  $y \in x$ , then  $y \in X_0 \cup X_{n-1}$ . Consider the bounded binary relation

$$\Phi(x, y) = \{(x, y) \mid (y \in X_0 \cup X_{n-1}) \wedge (x \in X_n) \wedge (y \in x) \wedge (x \in \mathcal{V})\}.$$

The domain of  $\Phi$  is  $\mathcal{V}$ . Let  $\{(a_1, b_1), \dots, (a_n, b_n)\} \subset \Phi$ . Since  $\mathcal{V}$  has the finite intersection property, there exists some  $b \in X_0 \cup X_{n-1}$  such that  $b \in a_1 \cap \dots \cap a_n$ . Thus  $\{(a_1, b), \dots, (a_n, b)\} \subset \Phi$ . Hence since we are in an enlargement, there exists  $a \in {}^*X_0 \cup {}^*X_{n-1}$  such that  $a \in {}^*V$  for each  $V \in \mathcal{V}$ . Consequently,  $\mu_{\mathcal{V}} \neq \emptyset$  and the proof is complete.

Note that if  $\mathcal{V}$  is a filter subbase, that the set obtained by taking finite intersections of members of  $\mathcal{V}$  does not contain the empty set and is closed under finite intersection.

**Theorem 6.3.** *Let  $X$  be standard set and  $\mathcal{V}$  any standard collection of subsets of  $X$ , which does not contain the empty set and which is closed under finite intersection. If internal  $A \subset \mu_{\mathcal{V}}$ , then there exists some \*-finite (and, hence, internal)  $\Omega \subset {}^*\mathcal{V}$  such that  $\sigma\mathcal{V} \subset \Omega$  and  $A \subset A_0 = \cap\{B \mid B \in \Omega\} \subset \mu_{\mathcal{V}}$ , and  $A_0 \in {}^*\mathcal{V}$ .*

Proof. First, from Theorem 6.2  $\mu_{\mathcal{V}} \neq \emptyset$ . Let  $B = \{V \mid (V \in {}^*\mathcal{V}) \wedge (A \subset V)\}$ . Then  $B$  is an internal subset of  ${}^*\mathcal{V}$ . Extending Theorem 4.3.4 [3], we know that there exists a \*-finite set  $B_0$  such that  $\sigma\mathcal{V} \subset B_0 \subset {}^*\mathcal{V}$ . Let  $\Omega = B \cap B_0$ . Since every internal subset of a \*-finite set is \*-finite,  $\Omega$  is \*-finite. Further,  $\sigma\mathcal{V} \subset \Omega$  and  $A \subset E$  for each  $E \in \Omega$ . By \*-transfer,  ${}^*\mathcal{V}$  is closed under \*-finite intersection. Hence  $A_0 = \cap\{B \mid B \in \Omega\} \in {}^*\mathcal{V}$ ,  $A \subset A_0$  and  $A_0 \subset \mu_{\mathcal{V}}$  and the proof is complete.

**Corollary 6.3.1.** *Let  $X$  be standard set and  $\mathcal{V}$  any standard collection of subsets of  $X$ , which does not contain the empty set and which is closed under finite intersection. Then  $\mu_{\mathcal{V}} = \cup\{G \mid (G \in {}^*\mathcal{V}) \wedge (G \subset \mu_{\mathcal{V}})\}$ .*

Proof. For every  $q \in \mu_{\mathcal{V}}$ , there is some  $G \in {}^*\mathcal{V}$  such that  $q \in G \subset \mu_{\mathcal{V}}$ .

A nonempty collection  $\mathcal{B}$  of subsets of  $X$  is a filter base, if  $\emptyset \notin \mathcal{B}$  and if  $A, B \in \mathcal{B}$ , then there exists some  $C \in \mathcal{B}$  such that  $C \subset A \cap B$ . A filter base is a filter subbase.

**Theorem 6.4.** *For a standard set  $X$ , let  $\mathcal{B}$  be a standard filter base defined on  $X$ . If internal  $A \subset \mu_{\mathcal{B}}$ , then there exists some \*-finite (and, hence, internal)  $\Omega \subset {}^*\langle\mathcal{B}\rangle$  such that  $\sigma\langle\mathcal{B}\rangle \subset \Omega$  and  $A \subset A_0 = \cap\{B \mid B \in \Omega\} \subset \mu_{\mathcal{B}}$ , and  $A_0 \in {}^*\langle\mathcal{B}\rangle$ .*

**Corollary 6.4.1.** *For a standard set  $X$ , let  $\mathcal{B}$  be a standard filter base defined on  $X$ . Then  $\mu_{\mathcal{B}} = \cup\{G \mid (G \in {}^*\langle\mathcal{B}\rangle) \wedge (G \subset \mu_{\mathcal{B}})\}$ .*

**Theorem 6.5.** *Given an internal set  $A$  and a standard filter subbase  $\mathcal{F}$  such that  $A \cap {}^*F \neq \emptyset$  for each  $F \in \mathcal{F}$ . Then  $A \cap \mu_{\mathcal{F}} \neq \emptyset$ .*

Proof. Consider the internal binary relation on  $\mathcal{B} = \langle\mathcal{F}\rangle$ .

$$\Phi = \{(a, b) \mid (b \in A) \wedge (b \in a) \wedge (a \in {}^*\mathcal{B})\}.$$

Suppose that  $\{(a_1, b_1), \dots, (a_n, b_n)\} \subset \Phi$  and  $a_i \in {}^\sigma\mathcal{B}$ . Since  $\mathcal{B}$  is closed under finite intersection, we have nonempty  $a \in \mathcal{B}$ , where  ${}^*a = {}^*b_1 \cap \dots \cap {}^*b_n$ . Thus there is some internal  $b' \in {}^*a$  such that  $(a_1, b'), \dots, (a_n, b') \subset \Phi$ . Since the cardinality of  $\mathcal{B}$  less than  $\kappa$ , then  $\Phi$  is, at least, concurrent on  ${}^\sigma\mathcal{B}$ . Thus there exists some  $q \in A \cap {}^*B$  for each  $B \in \mathcal{B}$ . This implies that  $A \cap \mu_{\mathcal{B}} = A \cap \mu_{\mathcal{F}} \neq \emptyset$  and the proof is complete.

**Theorem 6.6.** *Let  $\mathcal{B}$  be a standard filter base and suppose that an internal set  $\Lambda \subset {}^*\mathcal{B}$  has the property that  ${}^\sigma\mathcal{B} \subset \Lambda$ . Then there exists some internal  $A \in \Lambda$  such that  $A \subset \mu_{\mathcal{B}}$ .*

Proof. Consider the internal (bounded) binary relation

$$\Phi = \{(b, a) \mid ((b \in {}^*\mathcal{B}) \wedge (a \in \Lambda) \wedge (a \subset b))\}.$$

Let  $\{(b_1, a_1), \dots, (b_n, a_n)\} \subset \Phi$  and  $a_i \in {}^\sigma\mathcal{B}$ . Then there is some  $b' \in {}^\sigma\mathcal{B}$  such that  $b' \subset b_1 \cap \dots \cap b_n$ . But  $b' \in \Lambda$ . Hence,  $\{(b_1, b'), \dots, (b_n, b')\} \subset \Phi$ . The  $\kappa$ -saturation, there exists some internal  $A \in \Lambda$  such that  $A \subset {}^*B$  for each  $B \in \mathcal{B}$ . Hence,  $A \subset \mu_{\mathcal{B}}$  and the proof is complete.

**Theorem 6.7** *Let  $\mathcal{B}$  be a standard filter base and suppose that an internal set  $\Lambda \subset {}^*\mathcal{B}$  has the property that if  $G \in {}^*\mathcal{B}$  and  $G \subset \mu_{\mathcal{B}}$ , then  $G \in \Lambda$ . Then there exists some  $B \in \mathcal{B}$  such that  ${}^*B \in \Lambda$ .*

Proof. Assume the hypothesis and that there is no  $B \in \mathcal{B}$  such that  ${}^*B \in \Lambda$ . Then the internal set  ${}^*\mathcal{B} - \Lambda \subset {}^*\mathcal{B}$  satisfies the hypothesis for the “ $\Lambda$ ” of Theorem 6.6. Thus there exists some  $A \in {}^*\mathcal{B} - \Lambda$  such that  $A \subset \mu_{\mathcal{B}}$ . But from the hypothesis of this theorem, such an  $A \in \Lambda$ . This contradiction complete the proof.

In order to obtain a significant result that characterizes S-continuity, we need the following additional fact.

**Theorem 6.8** Consider standard  $X$ . Let internal  $\mathcal{B} \subset {}^*F({}^*X)$ , where  ${}^*F({}^*X)$  is the set of all  ${}^*$ -finite subsets of  ${}^*X$ . Suppose that whenever  $E \in {}^*F({}^*X)$  and  ${}^\sigma X \subset E \subset {}^*X$  then  $E \in \mathcal{B}$ . Then there exists  $F \in F(X)$  such that  ${}^*F \in \mathcal{B}$ .

Proof. To establish this, let  $\mathcal{F} = \{F \mid X - F \in F(X)\}$ . Then  $\mathcal{F}$  is a filter on  $X$ . This theorem is but an equivalent statement of theorem 6.6 in terms of the filter  $\mathcal{F}$ . Establishing this fact completes the proof.

We are now able to properly characterize the concept of S-continuity relative to standard pseudo-metric generated spaces.

**Theorem 6.9.** For standard  $X, Y$ , consider the pseudo-metric generated spaces  $({}^*X, {}^\sigma\Lambda), ({}^*Y, {}^\sigma\Pi)$ . An internal  $f: {}^*X \rightarrow {}^*Y$  is S-continuous at  $q \in {}^*X$  if and only if for each  $\epsilon \in \mathbb{R}^+$  and each  ${}^*\pi \in {}^\sigma\Pi$  there exists a finite set  ${}^*\lambda_i, 1 \leq i \leq n$  and positive  $\delta \in \mathbb{R}^+$  such that whenever  $x \in {}^*X$  and  ${}^*\lambda_i(x, q) < \delta$  for each  $i, 1 \leq i \leq n$ , then  ${}^*\pi(f(x), f(q)) < \epsilon$ .

Proof.  $\Rightarrow$  First, suppose that  $f$  is S-continuous at  $q \in {}^*X, {}^*\pi \in {}^\sigma\Pi$  and let  $\epsilon \in \mathbb{R}^+$ . For any internal binary relation  $A$ , let  $D(A)$  denote the internal domain and  $R(A)$  the internal range. Consider the internal set

$$T(\epsilon) = \{K \mid (\emptyset \notin D(K)) \wedge (\emptyset \notin R(K)) \wedge (K \in {}^*F({}^*\Lambda \times {}^*\mathbb{R}^+)) \wedge (\forall \lambda \forall \delta \forall x \\ ((\lambda \in D(K)) \wedge (\delta \in R(K)) \wedge (x \in {}^*X) \wedge (\lambda(x, q) < \delta) \rightarrow \\ ({}^*\pi(f(x), f(q)) < \epsilon)))\}.$$

By  $\kappa$ -saturation, we know that there exists a  ${}^*$ -finite  $K_0 \subset {}^*\Lambda \times {}^*\mathbb{R}^+$  such that  ${}^\sigma(\Lambda \times \mathbb{R}^+) \subset K_0$ . Thus  $\emptyset \neq T(\epsilon)$ . Further, suppose that  $K_1 \in {}^*F({}^*\Lambda \times {}^*\mathbb{R}^+)$  and  ${}^\sigma\Lambda \times \mathbb{R}^+ \subset K_1$ . Then  $D(K_1) = G_1 \in {}^*F({}^*\Lambda), {}^\sigma\Lambda \subset G_1$  and  $R(K) = H_1 \in {}^*F(\mathbb{R}^+), \mathbb{R}^+ \subset H_1$ . Now  $x \approx q$  implies that  ${}^*\lambda(x, q) < \delta$  for each  ${}^*\lambda \in {}^\sigma\Lambda$  and each  $\delta \in \mathbb{R}^+$ . But, S-continuity implies that  ${}^*\pi(x, q) < \epsilon$ . Hence  $K_1 \in T(\epsilon)$ . Since  $T(\epsilon)$  is internal, then Theorem 6.8 implies that there exists standard  $K'$  such that  $K' \in F(\Lambda \times \mathbb{R}^+)$  and such that  ${}^*K' \in T(\epsilon)$ . Consequently, there is positive  $n \in \mathbb{N}$  and  $\lambda_i \in D(K') \subset \Lambda$  when  $1 \leq i \leq n$  and a positive  $m$  and  $\delta_j \in R(K') \subset \mathbb{R}^+$  when  $1 \leq j \leq m$  and for each  $x \in X$ , if for each  $i$  and for each  $j$   $\lambda_i(x, q) < \delta_j$ , then  ${}^*\pi(f(x), f(q)) < \epsilon$ . Now simply consider  $\delta = \min\{\delta_1, \dots, \delta_m\}$  and  $\Rightarrow$  holds.

$\Leftarrow$  Suppose that  $f$  is not S-continuous at  $q$ . Then there exists some  $x_0 \in \mu_{{}^\sigma\Lambda}(q) = \cap\{\mu_{{}^*\pi}(q) \mid {}^*\pi \in {}^\sigma\Pi\}$ , such that  $f(x_0) \notin \mu_{{}^*\Pi}(f(q))$ . Thus there is some  $\pi \in \Pi$  such that  $f(x_0) \notin \mu_\pi(f(q))$ . Let standard  $\epsilon = \min\{1, \text{st}(\pi(f(x_0), f(q)))/2\}$  if  $\pi(f(x_0), f(q)) \in \mathcal{O}$ , in which case  $\epsilon > 0$  since  $\text{st}(\pi(f(x_0), f(q)))/2 \neq 0$ . Otherwise

take  $\epsilon = 1$ . Now for all standard  $\delta > 0$ , we have that for each  ${}^*\lambda \in {}^\sigma\Lambda$ ,  $\lambda(x_0, q) < \delta$ , but  ${}^*\pi(f(x_0), f(q)) \geq \epsilon$ . The proof is complete.

**Corollary 6.9.1.** *For standard  $X, Y$ , consider the pseudo-metric generated spaces  $({}^*X, {}^\sigma\Lambda)$ ,  $({}^*Y, {}^\sigma\Pi)$ . An extended standard map  ${}^*f: {}^*X \rightarrow {}^*Y$  is S-continuous at  ${}^*p \in {}^\sigma X$  if and only if  $f: (X, \Lambda) \rightarrow (Y, \Pi)$  is continuous at  $p \in X$ .*

Notice that the  $\delta$ s and  $\epsilon$ s that appear on Theorem 6.9 are standard real numbers. This theorem shows that close relationship between the concept of S-continuity and the concept of continuity. For the only difference within our  $\kappa$  saturated model between these two concepts when viewed from the external nonstandard physical world is that one hand the  $x$ s are members of  ${}^*X$  while on the other they elements of  $X$ .

One of the major facts about S-continuous functions is found in Theorem 1.1 in [4, p. 805.] As pointed out, for this theorem, the topological space  $X$  need not be compact and the first two parts of the theorem hold. In this theorem, the term microcontinuous is equivalent to S-continuous. For simplicity of notation, the sets  $X, Y$  are considered as a subset of the ground set that is used to generate our ultraproduct structure.

**Theorem 6.10.** *Suppose that you have the topological spaces  $X, Y, Y$  regular Hausdorff, where topological monads are defined at standard points, and an internal  $f: {}^*X \rightarrow {}^*Y$ . At  $p \in X$ , let  $f$  be S-continuous and suppose that there exists some  $r \in Y$  such that  $f({}^*p) \approx r$ . Then any function  $F: X \rightarrow Y$  such that  $F(p) = \text{st}(f({}^*p))$  is continuous at  $p$  in the topological sense. Further, if  $q \approx {}^*p$ , then  $f(q) \approx {}^*F({}^*p)$ . [Note. The standard part operator as defined in this theorem, brings points all the way back to the original standard set  $Y$ . It is not significant that this operator can be considered as defined on  $Z \subset {}^*Y$  and  $\text{st}(Z) \subset {}^\sigma Y$ .]*

Theorem 6.10 shows, in all generality, how S-continuity in general topological spaces leads directly to a standard function because the internal function ignores infinitesimal discontinuities. If the space  $Y$  is not be Hausdorff, then, by the Axiom of Choice, continuous function(s) can still be constructed.

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**Definition 6.8 (S-convergence)** Let internal sequence  $s: {}^*\mathbb{N} \rightarrow (X, \Lambda)$ , where  $(X, \Lambda)$  is a pseudo-metric generated space. Then  $s$  S-converges to  $q \in X$  if  $s_\omega \in \mu_\Lambda(q)$  for each infinite  $\omega \in \mathbb{N}_\infty$ .

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**Theorem 6.11.** *If, for pseudo-metric generated spaces, internal  $f: (X, \Lambda) \rightarrow (Y, \Pi)$  is S-continuous at  $q \in X$  and the internal sequence  $s: {}^*\mathbb{N} \rightarrow (X, \Pi)$  S-converges to  $q$ , then the internal sequence  $f(s)$  S-converges to  $f(q)$ .*

Proof. Suppose that the internal sequence  $s$  S-converges to  $q \in X$ . Let  $\omega \in \mathbb{N}_\infty$ . Then  $s_\omega \in \mu_\Lambda(q)$ . Thus,  $f(s_\omega) = (fs)_\omega \in \mu_\Pi(f(q))$  and the proof is complete.

The next theorem is similar to Theorem 6.9, relates S-convergence to standard approximations as well as to standard sequences.

**Theorem 6.12.** *For a pseudo-metric generated space  $(X, \Lambda)$ , an internal sequence  $s: {}^*\mathbb{N} \rightarrow X$ , S-converges to  $q \in X$  if and only if for each positive  $\epsilon \in \mathbb{R}$  each  $\lambda \in \Lambda$  there exists some  $M \in \mathbb{N}$  such that for each  $m > M$  (in  ${}^*\mathbb{N}$ ), it follows that  $\lambda(s_m, q) < \epsilon$ .*

Proof. ( $\Rightarrow$ ) Suppose the internal sequence  $s$  is S-convergent to  $q \in X$ . Let  $0 < \epsilon \in \mathbb{R}$ . Consider the internal set

$$m(\epsilon) = \{m \mid (m \in {}^*\mathbb{N}) \wedge (\forall n)((n \in {}^*\mathbb{N}) \wedge (n > m) \rightarrow \lambda(s_n, q) < \epsilon))\}.$$

From the definition of S-convergence  $\mathbb{N}_\infty \subset m(\epsilon)$ . However, the set  $m(\epsilon)$  has an internal range. Hence, there exists some standard  $m \in \mathbb{N}$  such that  $m \in m(\epsilon)$ . Consequently, the conclusion follows.

( $\Leftarrow$ ) Suppose that  $s$  is not S-convergent to  $q$ . Then there exists some  $s_\omega$ ,  $\omega \in \mathbb{N}_\infty$  such that  $s_\omega \notin \mu_\Lambda(q)$ . Hence, there is some  $\lambda \in \Lambda$  such that  $s_\omega \notin \mu_\lambda(q)$ . Let standard  $\epsilon = \min\{1, \text{st}(\lambda(s_\omega, q))/2\}$  if  $\lambda(s_\omega, q) \in \mathcal{O}$ , in which case  $\epsilon > 0$  since  $\text{st}(\lambda(s_\omega, q)/2) \neq 0$ . Otherwise take  $\epsilon = 1$ . Thus there exists some  $\omega \in \mathbb{N}_\infty$  such that and  $\lambda(s_\omega, q) \geq \epsilon$ . The proof is complete.

**Corollary 6.12.1.** *For a standard pseudo-metric space  $(X, \rho_X)$ , a standard sequence  $s: \mathbb{N} \rightarrow (X, \rho_X)$  is convergent to  $p \in X$  if and only if  ${}^*s$  is S-convergent to  $p$ .*

Now we need to define the limited points for internal pseudo-metrics. For an internal pseudo-metric  $\rho_X$ , you have a set of limited points per  $p \in X$ .

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**Definition 6.9 (Limited Points for Pseudo-metric Generated Spaces)**

Let  $(X, \Lambda)$  be a pseudo-metric generated space. Let  $q \in X$  Then  $\mathcal{O}_\Lambda(q) = \{x \mid (x \in X) \wedge \forall \lambda((\lambda \in \Lambda) \rightarrow \lambda(x, q) \in \mathcal{O})\}$ .

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**Theorem 6.13.** *Let  $({}^*X, \Lambda)$  be a pseudo-metric generated space. Suppose that each  $\lambda \in \Lambda$  is determined by an internal pseudo-norm  $\|\cdot\|_\lambda$  and that  ${}^*X$  is an internal linear space over  ${}^*\mathbb{R}$ . Suppose that for each  $p \in X$  and each  $\|\cdot\|_\lambda$ ,  $\|p\|_\lambda \in \mathcal{O}$ . Then for each  $p \in X$ ,  $\mathcal{O}_\Lambda(p) = \mathcal{O}_\Lambda(\mathbf{0}) = \{x \mid (x \in {}^*X) \wedge \forall \lambda ((\lambda \in \Lambda) \rightarrow (\|x\|_\lambda \in \mathcal{O}))\} = \mathcal{O}_X$ .*

Proof. Let  $x \in \mathcal{O}_\Lambda(p)$  for  $p \in X$ . Then for any  $\lambda \in \Lambda$  it follows that  $\|x\|_\lambda - \|p\|_\lambda \in \mathcal{O}$ . Hence,  $\|x\|_\lambda \in \mathcal{O}$  since  $\|p\|_\lambda \in \mathcal{O}$ .

Conversely, let  $x \in \mathcal{O}_\Lambda(\mathbf{0})$  and  $p \in X$ . Then for each  $\lambda \in \Lambda$ ,  $\|x\|_\lambda \in \mathcal{O}$ . But  $\|p\|_\lambda \in \mathcal{O}$  implies, since  $\|x - p\|_\lambda \leq \|x\|_\lambda + \|p\|_\lambda \in \mathcal{O}$ , that  $x \in \mathcal{O}_\Lambda(p)$ . The proof is complete.

**Corollary 6.6.1.** *Let the standard pseudo-metric  $\rho_X$  be defined on  $X \times X$  and be generated by a standard semi-norm  $\|\cdot\|_X$ . If  $r, p \in X$ , then  $\mathcal{O}_\rho(p) = \mathcal{O}_\rho(r) = \mathcal{O}_X$ .*

By Robinson's Theorem 4.3, the set  $T$  can be replaced by the set  ${}^*C^\infty \cap T$ . Hence, the usual practice has been to consider defining sets of internal semi-norms on the set  ${}^*C^\infty$  and consider the restriction such a set of internal semi-norms to the test space  $D$ . This collection can be composed of the nonstandard extensions of the customary set of standard semi-norms so that they correspond to the concept of Schwartz generalized functions and other standard types of generalized functions. However, it is also possible to broaden the collection of internal semi-norms in various ways. This is done in section 10.4 of reference [8].

## 7. Per-generalized Functions and S-continuity.

**Theorem 7.1.** *Let generalized function  $f \in T$  and standard  $p \in \mathbb{R}$ . Suppose that  $f$  is S-continuous at  $p$ . Then  $f(p) \in \mathcal{O}$ .*

Proof. Let  $f$  be S-continuous at  $p \in \mathbb{R}$ . Assume that  $f(p)$  is not limited. Without loss of generality, assume that  $f(p)$  is a positive infinite hyperreal number. Now for each  $x \approx p$ , it follows that  $f(x) > (1/2)f(p)$  since by S-continuity  $f(x) \approx f(p)$ . Note: For an infinite  $\Lambda$  and any  $\epsilon \in \mu(0)$ , the infinite  $\Lambda + \epsilon > (1/2)\Lambda$ . By the internal definition method, define that internal set

$$D = \{x \mid (x > 0) \wedge (x \in {}^*\mathbb{R}) \wedge (|x - p| > 0 \rightarrow (f(x - p) > (1/2)f(p)))\}.$$

Since  $D$  contains all of the positive infinitesimals, then by a modified 10.1.1 in [3],  $D$  contains a standard positive  $a$ . Consider the standard interval  $[p - a, p + a]$ . It is not difficult to construct a non-negative  $h \in \mathcal{D}$  such that  $h(x) = 1$ , for each  $x$  such that  $|x| \leq (1/2)a$  and  $h(x) = 0$  for  $|x| \geq a$ . Now let  $g = h(x - p)$ . Since  $g \in \mathcal{D}$ ,

$${}^*\int_{-\infty}^{\infty} f * g = {}^*\int_{p-a}^{p+a} f * g > (1/2)af(p).$$

The results follows from this contradiction.

**Corollary 7.1.1.** *Let internal  $f \in T$  and standard  $p \in \mathbb{R}$ . Suppose that  $f$  is  $S$ -continuous at  $p$ . Then any standard function  $F: \mathbb{R} \rightarrow \mathbb{R}$  such that  $F(p) = \text{st}(f(p))$  is continuous at  $p$  and  $f(\mu(p)) \subset \mu(F(p))$ .*

---

**Theorem 7.2.** *If  $f \in T$  is  $S$ -continuous at each  $p \in \mathbb{R}$ , then  $F(p) = \text{st}(f(p))$  is continuous on  $\mathbb{R}$ .*

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{**Remark.** If the function  $f \in T$  satisfies the requirements in Theorem 7.2, then  $f(\mathcal{O}) \subset \mathcal{O}$ . If  $f$  is also a surjection, then  $f(\mathcal{O}) = \mathcal{O}$ .}

**Example 7.1** We show that the converse of Theorem 7.2 does not hold. First, consider for any given standard  $a > 0$ , the set  $A = [-a, 0) \cup (0, a]$ , and the standard function defined by

$$f(x) = \begin{cases} 1, & x \in A \\ 0, & x \in \mathbb{R} - A. \end{cases}$$

Then  $f^2$  is Riemann integrable on any real  $[c, d]$ ,  $c \leq d$  and since  $\sup\{f^2(x) \mid x \in \mathbb{R}\} = 1$ ,  $\int_c^d f^2 \leq (d - c)$ . It follows from this that  $|\int_{-\infty}^{\infty} fg| \leq |\int_{-h}^h g| \in \mathbb{R}$  for some  $h \in \mathbb{R}$ . Now consider the internal function  $k$  obtained by means of the definition for  $f$  but let positive  $a = \epsilon \in \mu(0)$ . Selecting  $c, d \in \mathcal{O}$ , it follows that  $k \in T$ . Notice that for each  $p \in \mathbb{R}$ ,  $k(p) = 0$ . Thus  $K(p) = \text{st}(k(p)) = 0$  is a continuous function on  $\mathbb{R}$ . Let  $x = \epsilon/2 \in \mu(0)$ . Then  $k(x) = 1 \not\approx 0$ . Thus  $k$  is not  $S$ -continuous at  $x = 0$ .

One of the major advantages in using the nonstandard equivalence class method is that various members of a pre-generalized function  $\alpha$  can be specifically analyzed. This is not the case for the standard approaches to this subject.

**Theorem 7.3.** *If  $f \in \alpha$  is  $S$ -continuous at each  $p \in \mathbb{R}$ , then, for the continuous function  $F$  defined by  $F(p) = \text{st}(f(p))$ ,  ${}^*F \in \alpha$ .*

Proof. Since  ${}^*F \in T$ , it follows that  ${}^*F \in \alpha$  if and only if  ${}^*\int_{-\infty}^{\infty} (f - {}^*F) {}^*g \approx 0$  for all  $g \in \mathcal{D}$ . From the definition of  $g$ , there is  $c \in \mathbb{R}$  such that

$${}^*\int_{-\infty}^{\infty} (f - {}^*F) {}^*g = {}^*\int_{-c}^c (f - {}^*F) {}^*g.$$

Since  $[-c, c]$  is compact, if  $x \in {}^*[-c, c]$ , then there exists some  $p \in [-c, c]$  such that  $x \approx p$ . From  $S$ -continuity, standard part operator properties, and continuity,  $f(x) \approx f(p) \approx F(p) \approx {}^*F(x)$  implies that  $f(x) \approx {}^*F(x)$ . Consequently,  $|f(x) - {}^*F(x)| \approx 0$ .

Consider the internal set  $A = \{y \mid (y \in {}^*\mathbb{R}) \wedge \exists x((x \in {}^*[-c, c]) \wedge (y = |f(x) - {}^*F(x)|))\}$ . Since  $[-c, c]$  is compact,  $F([-c, c]) = [a, b]$  implies that  ${}^*F({}^*[-c, c]) = {}^*[a, b]$ . Further, from above,  $f({}^*[-c, c]) \subset [-c-1, c+1]$ . Consequently, the internal set  $A$  is  ${}^*$ -bounded. Recalling that the  ${}^*\sup = \sup$ , it follows by  ${}^*$ -transfer, that  $\sup A \in {}^*\mathbb{R}$ . But for each positive  $r \in \mathbb{R}$  and each  $y \in A$ ,  $0 \leq y < r$ . Hence, the  $\sup A$  is not a positive real number. Again by  ${}^*$ -transfer and the fact that  $A$  is internal,  $\sup A \in {}^*\mathbb{R}$ . Thus  $\sup A \approx 0$ . But,

$$\left| {}^*\int_{-c}^c (f - {}^*F) {}^*g \right| \leq \sup\{|f(x) - {}^*F(x)| \mid x \in {}^*[-c, c]\} \left( \int_{-c}^c |g| \right).$$

Hence,  ${}^*\int_{-c}^c (f - {}^*F) {}^*g \approx 0$  and the proof is complete.

**Corollary 7.3.1.** *If  $f \in \alpha$  is S-continuous at each  $p \in \mathbb{R}$ , then  $\alpha$  is a  $\mathcal{D}'$ -pre-generalized function.*

*Proof.* The  ${}^*$ -continuous function  ${}^*F \in \alpha$ . Consider a sequence  $\{g_n\} \subset \mathcal{D}$  such that  $g_n(x) = 0$  for all  $n$  and for all  $x$  such that  $|x| > c \in \mathbb{R}$ . Further suppose that  $g_n \rightarrow 0$  uniformly on for all  $x$  such that  $|x| \leq c$ . Then  $\lim_{n \rightarrow \infty} {}^*F[g_n] = \text{st}({}^*(\lim_{n \rightarrow \infty} \langle F, g_n \rangle)) = \text{st}({}^*\langle F, \lim_{n \rightarrow \infty} g_n \rangle) = \text{st}(0) = 0$ .

The existence of a function  $f$  in a pre-generalized function that is S-continuous at various members of  ${}^*\mathbb{R}$  seems to be of some significance. Indeed, the standard part of members of a pre-generalized function that are S-continuous at the same point in  $\mathbb{R}$  cannot be distinguished one from another at that point.

**Theorem 7.4.** *Suppose that  $f, h \in \alpha$  and that  $f$  and  $h$  are S-continuous at  $p \in \mathbb{R}$ . Then  $\text{st}(f(p)) = \text{st}(h(p))$  and if  $x \approx p$ , then  $f(x) \approx h(x)$ .*

*Proof.* We know from Theorem 6.1 that  $f(p), h(p) \in \mathcal{O}$ . Thus  $\text{st}(f(p)), \text{st}(h(p)) \in \mathbb{R}$ . All we need to do is to show that  $f(p) \approx h(p)$ . Since  $f, h \in \alpha$ , the function  $k = f - h \in T_0$ . Suppose that  $f(p) - h(p)$  is not infinitesimal. Without loss of generality, in this case, consider  $k(p) = f(p) - h(p) > r > 0$ ,  $r \in {}^*\mathbb{R} - \mu(0)$ . As in the proof of Theorem 7.1, there is a  $g \in \mathcal{D}$ , such that  $k[g]$  is not infinitesimal; a contradiction. Thus  $f(p) \approx h(p)$ . Obviously, if  $x \in p$ , S-continuity implies that if  $x \approx p$ , then  $f(x) \approx f(p) \approx h(p) \approx h(x)$ . Hence  $f(x) \approx h(x)$  and the proof is complete.

Due to Theorem 7.4, a pre-generalized function  $\alpha$  that contains an internal function  $f$  that is S-continuous at  $p \in \mathbb{R}$  can be considered a function itself.

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**Definition 7.1.** ( $\alpha$  as a Function on  $\mathbb{R}$ .) If  $f \in \alpha$  and  $f$  is S-continuous at  $p \in \mathbb{R}$  let  $\alpha(p) = \text{st}(f(p)) = F(p)$ .

---

Theorem 7.4. shows that definition 7.1 leads to a function-like pre-generalized function  $\alpha(p)$ . However, one other aspect of this  $\alpha$  function concept needs to be addressed. How unique is such a  $\alpha$  function with respect to members of  $T$ ?

**Theorem 7.5.** Suppose that  $f \in \alpha$  is S-continuous at each  $p \in \mathbb{R}$ . Let  $h \in T$  be S-continuous at each  $p \in \mathbb{R}$  and  $h(p) = f(p)$ . Then  $h \in \alpha$ .

Proof. (Notice that simply because  $h(p) = f(p)$  at the standard points does not imply the functions are equal at the nonstandard points.) What is needed is to show that  $\langle (f - h), {}^*g \rangle \approx 0$  for each  $g \in \mathcal{D}$ . Consider  $f - h \in T$ . Let  $g \in \mathcal{D}$ . Then there exists some positive  $c \in \mathbb{R}$  such that  $g(x) = 0$  for  $|x| \geq c$ . Since  $f(\mu(a)) \subset \mu(f(p))$  and  $h(\mu(a)) \subset \mu(h(a))$  and  $f(a) = h(a)$ ,  $(f - h)(\mu(a)) \subset \mu(0)$ . Consider a specific standard positive  $\epsilon$  and the internal set  $D(\epsilon) =$

$$\{y \mid (y \in {}^*\mathbb{R}) \wedge \forall x((x \in {}^*\mathbb{R}) \wedge (|x - a| < \epsilon) \rightarrow (|f(x) - h(x)| < y))\}.$$

Then  $D(\epsilon)$  contains all of the positive infinitesimals. As in the proof of Theorem 7.1, there exists a standard positive  $\eta_a$  such that  $|f(x) - h(x)| < \epsilon$  for each  $x$  such that  $|x - a| < \eta_a$ . For a fixed  $\epsilon$ , for each  $a \in [-c, c]$ , the positive standard  $\eta_a$  exists. The set of open intervals  $\{U_a \mid (a \in [-c, c]) \wedge (U_a = \{x \mid (a - \eta_a < x < a + \eta_a)\})\}$  is an open cover of compact  $[-c, c]$  and as such there exists a finite subset  $\{U_1, \dots, U_n\}$  that covers  $[-c, c]$ . Also,  $|f(x) - h(x)| < \epsilon$  for each  $x \in {}^*U_i$ . Since the set  $\{U_i\}$  is nonempty and finite it can be adjusted by moving in the end points if necessary (remove some of the open intervals) so that the shortened open intervals thus obtained have the property that no point in  $[-c, c]$  is contained in more than two such intervals and such that the sum of the lengths is not greater than say  $4c$ . Denote these adjusted intervals by  $H_i$ . Again  $|f(x) - h(x)| < \epsilon$  for each  $x \in {}^*H_i$ . This same procedure can be done for any arbitrary positive  $\epsilon$ .

Now let  $\lambda = \max |g(x)|$ . It is a known fact that  $g(x) = g_1(x) + \dots + g_n(x)$ , where each  $g_i \in \mathcal{D}$  and, for  $i = 1, \dots, n$ ,  $g_i(x) = 0, x \in \mathbb{R} - H_i$  and  $|g_i(x)| \leq \lambda, x \in \mathbb{R}$ . Hence

$$\left| {}^*\int_{-\infty}^{\infty} (f - h) {}^*g \right| = \left| \sum_{i=1}^n {}^*\int_{H_i} (f - h) {}^*g_i \right| \leq \sum_{i=1}^n {}^*\int_{H_i} \epsilon \lambda \leq 4c\epsilon\lambda.$$

But, as pointed out, positive  $\epsilon$  is arbitrary. Thus

$$\left| \int_{-\infty}^{\infty} (f - h) * g \right| \approx 0$$

and the proof is complete.

**Corollary 7.5.1.** *Suppose that  $f, h \in T$  are S-continuous at each  $p \in \mathbb{R}$  and  $f(p) = h(p)$  for  $p \in \mathbb{R}$ . Then there is a unique  $\alpha$  such that  $f, h, {}^*F \in \alpha$ ,  $F = H$ ,  $F$  is continuous on  $\mathbb{R}$  and  $\alpha(p) = \mathbf{st}(f(p)) = \mathbf{st}(h(p)) = F(p)$  for each  $p \in \mathbb{R}$ .*

Another aspect of this idea of S-continuity and pre-generalized functions is the observation that defining  $\alpha(p) = \mathbf{st}(f(p))$ , for some  $f \in \alpha$  that is S-continuous at  $x = p$ , is independent of the S-continuous function contained in  $\alpha$  by Theorem 7.4. Thus many  $\alpha$  can be considered as functions on large domains of real numbers. For example, the function  $d$ , in example 4.1, is in  $T$  and is S-continuous at every nonzero real number. Thus using this function the Dirac  $\mathcal{D}'$ -pre-generalized function  $\delta$  is a function  $\delta(x)$  for all nonzero real numbers. Then we have certain algebraic properties associated with the  $\alpha \in T/T_0$  that are functions at certain standard points.

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**Definition 7.2 ( $\alpha$  as a Function)** Call a per-generalized function  $\alpha$  a function at  $p \in \mathbb{R}$ , with value  $\alpha(p)$ , if there is an  $f \in \alpha$  that is S-continuous at  $p$  and let  $\alpha(p) = \mathbf{st}(f(p))$ .

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**Theorem 7.6.** *If  $\alpha, \beta$  are functions at  $p$ , then  $\gamma = \alpha \pm \beta$  is a function at  $p$  and  $\gamma(p) = \alpha(p) \pm \beta(p)$ .*

Proof. From the hypothesis, there exist two internal  $f, h \in T$  such that  $f, h$  are S-continuous at  $p$  and  $f \in \alpha, h \in \beta$ . Hence, the internal function  $f \pm h \in T$  is S-continuous at  $p$  and  $f \pm h \in \gamma$ . Since  $\mathbf{st}(f(p) \pm h(p)) = \mathbf{st}(f(p)) \pm \mathbf{st}(h(p))$ , it follows that  $\gamma(p) = \alpha(p) \pm \beta(p)$  and this completes the proof.

From Theorem 4.3, we know that every member of  $\mathcal{F}$  is generated by a member of  ${}^*C^\infty$ . This leads to the concept of the derivative of a generalized function. Unfortunately, if you want to define multiplication for the functions in  $T$  and use the usual concept that multiplication is independent of the member chosen from one or both of two pre-generalized functions  $\alpha$  and  $\beta$ , then multiplication must be restricted to certain per-generalized functions. Theorem 7.4 states that if  ${}^*f, {}^*h \in {}^\sigma C^\infty$  and  ${}^*f, {}^*h \in \alpha$ , then, since  ${}^*f, {}^*h$  are S-continuous at each  $p \in \mathbb{R}$ ,

$\text{st}(*f(p)) = f(p) = \text{st}(*h(p)) = h(p)$ . Thus  $f = h$ . Whenever possible such a unique member of  ${}^\sigma C^\infty$  can be used to generate a unique product-like per-generalized function.

**Theorem 7.7.** *Let  $*f \in {}^\sigma C^\infty$  and  $*f \in \alpha$  (hence, a  $\mathcal{D}'$ -pre-generalized function). Suppose that  $h, k \in \beta$ . Then there exists a pre-generalized function  $\gamma$  such that  $(*f)h, (*f)k \in \gamma$ .*

Proof. Notice that if  $*f \in {}^\sigma C^\infty$  and  $*g \in {}^\sigma \mathcal{D}$ , then  $*f * g \in {}^\sigma \mathcal{D}$ . From Theorem 3.1 (c), (e),  $(*f)h, (*f)k \in T$ . Thus there is a pre-generalized function  $\gamma$  such that  $(*f)h \in \gamma$ . Consider  $\langle (*f)h - (*f)k, *g \rangle$  for any  $g \in \mathcal{D}$ . Then  $\langle (*f)h - (*f)k, *g \rangle = \langle h - k, (*f) * g \rangle \approx 0$ .

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**Definition 7.3. (Customary Products)** Let  $*f \in {}^\sigma C^\infty$ ,  $*f \in \alpha$ . Then the pre-generalized function  $\gamma$  such that for each  $h \in \beta$ ,  $(*f)h \in \gamma$  is the customary product of  $\alpha$  and  $\beta$  and is denoted by  $\gamma = \alpha\beta$ .

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**Theorem 7.8.** *Let  $\{ *f, *h \} \subset {}^\sigma C^\infty$  and  $*f \in \alpha$ ,  $*h \in \beta$ . Then for  $\gamma = \alpha\beta$  it follows that  $\gamma' = \alpha'\beta + \alpha\beta'$ .*

Proof. We know that  $(*f * h)' \in \gamma'$ ,  $*f' \in \alpha'$ ,  $*h' \in \beta'$ . Of course,  $(*f * h)' = (*f') * h + (*f) * h'$ . Notice that  $(*f')h \in \alpha'\beta$ ,  $(*f) * h' \in \alpha\beta'$ . Hence,  $(*f') * h + (*f) * h' \in \alpha'\beta + \alpha\beta'$ . Then we have that  $((f) * h)' \in \gamma'$ . But the pre-generalized functions are equivalence classes. Thus  $\gamma' = \alpha'\beta + \alpha\beta'$  and this completes the proof.

**Lemma 7.1.** *Let  $f: {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$  and  $h: {}^*\mathbb{R} \rightarrow {}^*\mathbb{R}$  be  $S$ -continuous at  $p \in \mathbb{R}$ . Then the product function  $k(x) = f(x)h(x)$  is  $S$ -continuous at  $p$ .*

Proof. Let  $q \approx p$ . Then  $k(q) - k(p) = f(q)h(q) - f(p)h(p) = f(q)(h(q) - h(p)) + (f(q) - f(p))h(p)$ . Notice that  $f(q), h(p) \in \mathcal{O}$  and  $(h(q) - h(p)), (f(q) - f(p)) \in \mu(0)$ . Hence,  $k(q) - k(p) \in \mu(0)$ . The result is complete.

**Theorem 7.9.** *Let  $*f \in {}^\sigma C^\infty$  and  $*f \in \alpha$ . Assume that  $\beta$  is a function at  $p \in \mathbb{R}$ . Let  $\gamma = \alpha\beta$ . Then  $\gamma$  is a function at  $p \in \mathbb{R}$  and  $\gamma(p) = \alpha(p)\beta(p)$ .*

Proof. Since  $f$  is  $S$ -continuous at  $p$ ,  $f(p) = \alpha(p)$ . Further, there is some  $h \in \beta$  that is  $S$ -continuous at  $p$  and  $\beta(p) = \text{st}(h(p))$ . Consider  $k = fh \in \gamma$ . From Lemma 7.1,  $\gamma$  is a function at  $p$  and since  $\text{st}(f(p)h(p)) = \text{st}(f(p))\text{st}(h(p))$  the result follows.

Another result similar to Theorem 7.5, but not as definitive, can be obtained using the Theorem 7.5 method.

**Theorem 7.10.** *Let  $f \in T$  be  $S$ -continuous at each  $x$  such that  $a \leq x \leq b$ ,  $a < b$ ,  $a, b \in \mathbb{R}$  and  $\alpha$  a function at each  $x$  such that  $a \leq x \leq b$ ,  $a < b$ ,  $a, b \in \mathbb{R}$ . Then for every  $g \in \mathcal{D}$  such that the support of  $g$  is a subset of  $[a, b]$ , and every  $h \in \alpha$  it follows that  $f[g] = h[g]$ .*

Proof. Let  $f \in \beta$ . What is needed is to show that  $\langle (f - h), {}^*g \rangle \approx 0$  for each  $g \in \mathcal{D}$  such that the support of  $g$  is a subset of  $[a, b]$ . Taking the  $c$  in the proof of Theorem 7.5 such that  $[-c, c] \subset [a, b]$ , the proof is exactly the same as that of Theorem 7.5.

Any pre-generalized function that is a function on  $\mathbb{R}$  is an  $\mathcal{D}'$ -per-generalized function. This idea can be extended to the  $k$  derivatives of per-generalized functions.

**Theorem 7.11.** *Suppose that  $\alpha$  is a function on  $\mathbb{R}$ . Then for each positive  $k \in \mathbb{N}$ , the pre-generalized function  $\alpha^{(k)}$  is a  $\mathcal{D}'$ -pre-generalized function.*

Proof. From the hypothesis, there is a function  $F$  continuous on  $\mathbb{R}$  such that  ${}^*F \in \alpha$ . From Theorem 4.3, there is some  $h \in {}^*C^\infty$  such that  $h \in \alpha$  and, from the Definition of  $\alpha^{(k)}$ ,  $h^{(k)} \in \alpha^{(k)}$ . Let  $\{g_n\}$  be a sequence of members of  $\mathcal{D}$  such that  $g_n(x) = 0$  for all  $n$  and all  $x$  such that  $|x| > c$ ,  $c > 0$  and  $\{g_n^{(k)}\}$  converges uniformly in  $x$  for all positive  $k$ . All that is needed is to show that the sequence  $h^{(k)}[g_n]$  converges to zero. We know by parts integration that  $h^{(k)}[g_n] = (-1)^k h[g_n^{(k)}]$ . Further,  ${}^*F[g_n^{(k)}] = h[g_n^{(k)}]$ . From uniform convergence,  $\lim_{n \rightarrow \infty} {}^*F[g_n^{(k)}] = \lim_{n \rightarrow \infty} h[g_n^{(k)}] = \lim_{n \rightarrow \infty} h^{(k)}[g_n] = 0$ . The result follows.

## 8. Generalizations and Beyond

Every theorem in the previous sections relative to pre-generalized functions, holds true if the domain  ${}^*\mathbb{R}$  is changed to an open domain  $\Omega \subset {}^*\mathbb{R}^m$ . [Note: Theorem 4.3 still holds in the sense that there exists  $f \in {}^*C^\infty(\mathbb{R}^m)$  that generates each pre-generalized function.] Depending upon the level of pre-generalized function differentiation desired, the test space  $\mathcal{D}$  can be enlarged or reduced. For example, if you only wish that for each  $\alpha$ , there exists  $\alpha^{(k)}$  where  $1 \leq k \leq m$ . Then the test spaces can be the set of all functions  $g: \mathbb{R} \rightarrow \mathbb{R}$  that have bounded support and are continuously differentiable of order  $m$ . In this case, all the previous theorems on pre-generalized functions, modified for this degree of differentiability, also hold. Then another approach that has proved to be accessible to nonstandard methods is the generalization due to Colombeau [8]. I mention that the new approach by Egorov [9] could lead to a very accessible nonstandard theory and needs to be fully

explored.

Using section 6 and an appropriate collection of internal semi-norms, the concepts of the S-limit and S-convergence, and the like, can now be applied to pre-generalized functions.

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